

Completeness of the isomorphism problem for separable C^* -algebras

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Received: 20 November 2014 / Accepted: 26 August 2015 /

Published online: 24 September 2015

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Abstract This paper studies the descriptive set-theoretical complexity of the isomorphism problem for separable C^* -algebras. We prove that the isomorphism problem for separable (simple, AI) C^* -algebras is complete in the class of orbit equivalence relations. This means that any isomorphism problem arising from a continuous action of a separable completely metrizable group can be reduced to the isomorphism of simple, separable AI C^* -algebras.

Mathematics Subject Classification 03E15 · 46L35 · 46A55

1 Introduction

Broadly speaking, a problem P in a class Γ is called *complete* in Γ if any other problem in Γ can be reduced to P . Complete problems typically appear

This research was partially supported by the NCN (the Polish National Science Centre) Grant no. 2012/05/D/ST1/03206, by the Foundation for Polish Science and by MNiSW (the Polish Ministry of Science and Higher Education) Grants no. 0435/IP3/2013/72 and N N201 418939.

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in logic and computer science, perhaps with the most prominent examples of NP-complete problems.

In the continuous setting, a descriptive set-theoretic complexity theory for problems arising as Borel and analytic equivalence relations on standard Borel spaces, has been developed by Kechris, Louveau, Hjorth and others [2, 38, 40, 41, 48] over the last 30 years. The classification problems arising in this setting are of the following form: given an analytic or Borel equivalence relation E on a standard Borel space X , decide whether two points in X are E -equivalent. Of great interest here are the equivalence relations given by Borel actions of separable completely metrizable (i.e. *Polish*) groups on standard Borel spaces or, equivalently, continuous actions of Polish groups on Polish spaces (see [8]). Typically, isomorphism problems arising in various areas of mathematics are easily translated into this language. The relative complexity is measured in terms of Borel reducibility: an equivalence relation E on X is *Borel reducible* to an equivalence relation F on Y if there is a Borel map $f : X \rightarrow Y$ such that $x_1 E x_2$ if and only if $f(x_1) F f(x_2)$ for every $x_1, x_2 \in X$. The meaning of this notion is that the function f , being computable (Borel), gives a way of reducing the problem of E -equivalence of points in X to that of F -equivalence of points in Y . We say that an equivalence relation E is *complete* in a class Γ of equivalence relations if it belongs to Γ and every relation F in Γ is Borel-reducible to E . For examples of complete analytic equivalence relations see [29, 56]. Two relations E and F are *bi-reducible* if E is Borel reducible to F and F is Borel reducible to E . For countable group actions, also a stronger notion (in the measure-theoretic context) is used: two group actions on standard Borel measure spaces X and Y are called *orbit equivalent* if there is a Borel isomorphism of X and Y which maps (a.e.) orbits to orbits (see [32]). We say that an equivalence relation is an *orbit equivalence relation* if it is bi-reducible with an equivalence relation induced by a Borel action of a Polish group. Descriptive set-theoretic complexity theory has enjoyed spectacular successes, for instance the striking results of Thomas [66, 67] on the relative complexity of isomorphism problems for torsion-free abelian groups or the results of Foreman, Rudolph and Weiss [31] on the conjugacy problem in ergodic theory.

The isomorphism problem for separable C^* -algebras has been studied since the work of Glimm in the 1960's and evolved into the Elliott program that classifies C^* -algebras via their K -theoretic invariants. Glimm's result [35], restated in modern language, implies that the isomorphism relation for UHF algebras is smooth (see [33, Chapter 5.4]). In the 1970's the classification has been pushed forward to AF algebras via the K_0 group [17]. The Elliott invariant, which consists of the groups K_0 and K_1 together with the tracial simplex and the pairing map, was conjectured (see [19, 24]) to completely classify all infinite-dimensional, separable, simple nuclear C^* -algebras. The conjecture has been verified for various classes of C^* -algebras, e.g. certain classes of

real rank zero algebras, AH algebras of slow dimension growth or separable, simple, purely infinite, nuclear algebras (modulo the universal coefficient theorem) [18, 21–23, 49, 50, 64] and there have been dramatic breakthroughs in the program, including the counterexamples to the general classification conjecture constructed by Rørdam [65] and Toms [69].

The classification program of separable C^* -algebras can be studied from the point of view of descriptive set-theoretic complexity theory (cf [20]). The framework here has been set up in 1996 by Kechris [47] and more recently by Farah, Toms and Törnquist [27, 28].

The following question has become famous.

Question 1.1 (Farah et al, [1, 14, 16, 26, 28]) What is the complexity of the isomorphism relation for separable (nuclear, simple) C^* -algebras?

The question is strictly connected with a long-standing conjecture (posed in the late 1990's) of Gao, Hjorth and Kechris.

Conjecture 1.2 (Hjorth [40], Gao–Kechris [34]) *The homeomorphism relation of compact metric spaces is a complete orbit equivalence relation.*

For some background on these problems the reader is advised to consult the lecture¹ of Farah at the ICM in Seoul [25].

In this paper we answer Question 1.1.

Theorem 1.3 *The isomorphism relation of separable C^* -algebras is complete in the class of orbit equivalence relations. In fact, already the isomorphism of separable simple AI C^* -algebras is a complete orbit equivalence relation.*

We would like to note here that very recently Zielinski [72] used the results of this paper (Theorem 1.4 below) to give a very elegant proof of Conjecture 1.2.

Question 1.1 appears on a 2008 problem list of Farah [26]. Since then, it has been a focus of intense research and several workshops have been devoted to this question (at AIM, BIRS, Luminy, Singapore, and others). Question 1.1 appears on a problem list supported by the American Institute of Mathematics [1, Problem 7.2] and on the list edited by Effros, Elliott, Farah and Toms [14, Problem (3)]. Also recently, it has been stated in several forms as [28, Problem 9.3], [28, Problem 9.7] (asked by Hjorth), [16, Question 4.1], [16, Question 4.2].

A lower bound on the complexity of the isomorphism of separable simple AI algebras has been obtained by Farah, Toms and Törnquist [28], who showed that the isomorphism of separable simple AI algebras is not classifiable by countable structures (see [33, Chapter 10]). The upper bound showing that it is an orbit equivalence relation has been proved recently by Elliott, Farah,

¹ Which also includes a discussion of the results of the present paper.

Paulsen, Rosenthal, Toms, and Törnquist [16] using the technique of Gao and Kechris [34].

One has to mention here that an important point in the K -theoretic classification of separable simple nuclear C^* -algebras is the functoriality of the invariants. While the theory of functorial reductions is still to be developed in the context of descriptive set theoretic complexity theory, it is worth noting that our proof of Theorem 1.3 shows features of functoriality. The details are discussed in Sect. 9.

Using the results of Thomsen [68], Farah, Toms and Törnquist [28] showed that the relation of affine homeomorphism of Choquet simplices is Borel reducible to the isomorphism relation of simple, separable, AI algebras. Theorem 1.3 will thus follow from the following.

Theorem 1.4 *The relation of affine homeomorphism of Choquet simplices is complete in the class of orbit equivalence relations.*

Gao and Kechris [34] and, independently, Clemens showed that the isometry relation of separable complete metric spaces is a complete orbit equivalence relation. Our proof of Theorem 1.4 reduces the relation of isometry of separable complete metric spaces to the affine homeomorphism of Choquet simplices.

The proof of Theorem 1.4 will be geometric. It will be based on two main new ideas connecting metrics on separable metric spaces with the affine geometry of Choquet simplices. In the first part of the proof we construct Choquet simplices using specific families of Lipschitz functions defined on a separable metric space. Here, we additionally incorporate an idea coming from the Katětov construction of the universal Urysohn space and use it in the analysis of the extreme boundary in our construction. In the second part, we introduce the *blow-up construction* that rigidifies the structure of a simplex with respect to a specific metric defined on its extreme points. This is later used to encode a metric into the affine structure of a simplex. The final construction is a fusion of these two methods.

It is worth noting here that Haydon [39] (see also [10, Theorem 29.9] and [13, Page 143]) showed that any Polish space is homeomorphic to the extreme boundary of a simplex. One should note, however, that this construction is not unique and, indeed, there are simplices with homeomorphic extreme boundaries which are not affinely homeomorphic (see [3, Page 119]). The general problem of determining a simplex from the structure of its extreme boundary is called the *Dirichlet extension problem* and is solved in [4]. In this paper, we take a different approach and, given a separable metric space, we construct a simplex whose extreme boundary only contains the metric space as a subspace. The advantage over Haydon's construction is that this construction is invariant under the isometry. Our simplices, which we call *S-extensions* of

metric spaces, seem to be different from most of the typical constructions of simplices appearing in the literature.

This paper is organized as follows. In Sect. 2 we recall some basic facts from convex analysis and Choquet's theory. In Sect. 3 we recall and slightly strengthen the results of Clemens, Gao and Kechris on the isometry of separable metric spaces. Section 4 contains some elementary back-and-forth constructions for building affine homeomorphism. The S -extension construction appears in Sect. 5. Sections 6 and 7 form a preparation for a coding construction of metrics. The main construction appears in Sect. 8. Section 9 contains some concluding remarks and questions.

2 Convex analysis and Choquet's theory

For basic concepts of convex analysis and Choquet's theory we refer the reader to [44, Chapter 15] and to [5, 6, 10, 62]. In this paper we use the terms *simplex* and *Choquet simplex* interchangeably and all convex compact sets that we consider are metrizable.

We consider the Hilbert cube $[0, 1]^{\mathbb{N}}$ as a compact convex subset of a locally convex topological vector space (e.g. ℓ_{∞} with the weak* topology) and with the standard metric on $[0, 1]^{\mathbb{N}}$ given by $d_{[0,1]^{\mathbb{N}}}(x, y) = \sum_n 2^{-n} |x(n) - y(n)|$. Given a set A in a locally convex vector topological space, we write $\text{conv}(A)$ for the closed convex hull of A . For a compact convex set C , we write $\text{ext}(C)$ for the set of extreme points of C . Given two convex compact sets C and D , we write $C \simeq D$ to denote that C and D are affinely homeomorphic. We write Δ^n for the n -dimensional simplex. The Poulsen simplex [63] is the unique (see [54]) metrizable Choquet simplex with dense extreme boundary.

Whenever we consider a metric on a topological space, we assume it is compatible with the topology. Although convex compact sets are typically considered only with the affine and topological structure, we will sometimes use metrics on convex compact sets in locally convex vector topological spaces. We will always assume that if C is a compact convex set with a metric d_C on it, then there is a bigger convex compact set D with $C - C \subseteq D$ and a norm $\|\cdot\|$ on D such that $d_C(x, y) = \|x - y\|$ for every $x, y \in C$. For convex compact subsets of the Hilbert cube, we can use the metric $d_{[0,1]^{\mathbb{N}}}$ and for subsets of \mathbb{R} the standard distance on \mathbb{R} . Note that the open balls in such metrics are convex.

Given an inverse system (S_n, π_n) of simplices (with $\pi_n : S_{n+1} \rightarrow S_n$ an affine continuous surjection) and $x \in \varprojlim S_n$, we write $x \restriction S_i$ (or even $x \restriction i$ if it does not cause confusion) for the image of x under the canonical projection map from $\varprojlim S_n$ to S_i . Sometimes, we use the notation $x \restriction S_i$ for $x \in S_j$ with $j > i$. It is worth noting here that, in general, it is not true that any continuous affine surjection is open and the exact characterization of when this happens has been given by Vesterstrøm [71]. The inverse limit of a system of simplices

is again a simplex [43] and Lazar and Lindenstrauss [52] proved that any (metrizable) simplex is an inverse limit of a sequence of finite-dimensional simplices. Recently, López-Abad and Todorčević [55] showed that a generic inverse limit of simplices is affinely homeomorphic to the Poulsen simplex. For more on the Poulsen simplex we refer the reader to [44, Chapter 15] and [6, Chapter 3 Section 7].

The analysis on convex compact sets is dual to the theory of order unit Banach spaces via the spaces of affine functions (see [6, Chapter 2] and [15]).

The standard Borel structure on the space of simplices was introduced in [28] and is based on the parametrization of simplices proved by Lazar and Lindenstrauss [52] and the duality to the order unit spaces. Alternately, one can also use the induced Borel structure from the space of compact subsets of the Hilbert cube (see [28, Section 4.1.4]) or of the Poulsen simplex. Note that for a convex compact set C , the space of compact convex subsets of C is closed in $K(C)$. The space of simplices contained in C is not closed but it is Borel in $K(C)$ [28, Lemma 4.7].

We will need the following simple observation about affine continuous functions on convex compact sets. It is proved by an obvious averaging argument.

Proposition 2.1 *Let C, D be convex compact sets in locally convex vector topological spaces and let d_C be a metric on C . If $f, g : D \rightarrow C$ are two affine continuous functions such that $d_C(f(e), g(e)) \leq \varepsilon$ for some $\varepsilon > 0$ and every $e \in \text{ext}(D)$, then $\|f - g\|_\infty \leq \varepsilon$.*

3 Isometry of perfect Polish spaces of bounded diameter

Recall that the Urysohn space \mathbb{U} [70] is the universal ultrahomogeneous separable metric space, and for every $r > 0$ the bounded Urysohn space \mathbb{U}_1 is its counterpart of diameter 1. Both these spaces are constructed in the same way and have finite isometry extension properties. Also, in the literature, \mathbb{U}_1 is referred to as the *Urysohn sphere* and, as noted by Melleray, it is isometric to the sphere of diameter 1 (around any point) in the Urysohn space. For more on the Urysohn space and sphere, see [61, Chapter 5] or [53, 59, 60].

The standard Borel space of separable complete metric spaces is identified with the space of closed subsets of the Urysohn space \mathbb{U} with its Effros Borel structure. Gao and Kechris [34], and independently Clemens [11, 12], proved that the isometry relation on the space of separable complete metric spaces is complete in the class of orbit equivalence relations.

Similarly, we consider the standard Borel space of separable complete metric spaces of diameter bounded by 1 as the space of closed subsets of the Urysohn sphere \mathbb{U}_1 .

For a Polish space X , write $F_p(X)$ for the space of perfect (nonempty) subsets of X . Note that if X is compact, then $F_p(X)$ is G_δ in $K(X)$, which

implies that for every Polish space X the set $F_p(X)$ is Borel in $F(X)$. Given this, the standard Borel space of perfect separable metric spaces of diameter bounded by 1 is identified with $F_p(\mathbb{U}_1)$.

The following result is essentially due to Gao, Kechris and Clemens, and can be proved by an easy modification of either the proof of Gao and Kechris [34] or of the proof of Clemens [11]. Below we present a nice and short proof suggested by the Referee.

Proposition 3.1 *The isometry relation of perfect spaces of diameter less than 1 is complete in the class of orbit equivalence relations.*

Proof Embed the Banach space $C(2^{\mathbb{N}})$, of continuous functions on the Cantor set, isometrically (as a metric space) into the Urysohn space \mathbb{U} . This embedding induces a Borel map Λ from the space of closed subspaces of $C(2^{\mathbb{N}})$ to $F_p(\mathbb{U})$ (as every Banach space is perfect). The map Λ is simply taking separable Banach spaces and viewing them as metric spaces. By the work of Melleray [58], the relation of linear isometry between separable Banach spaces is complete in the class of orbit equivalence relations. As every surjective isometry between Banach spaces is affine, the map Λ is a reduction from linear isometry of separable Banach spaces to the isometry of perfect metric spaces. So the latter is also complete in the class of orbit equivalence relations.

Finally, consider the map $(X, d_X) \mapsto (X, \frac{d_X}{1+d_X})$ and note that it is a reduction from the isometry relation of perfect metric spaces to the isometry relation of perfect metric spaces of diameter less than 1. \square

Gao and Kechris [34, Section 2D] show also that the isometry relation of subspaces of \mathbb{U} is bireducible with the equivalence relation induced by the action of $\text{Iso}(\mathbb{U})$ on $F(\mathbb{U})$. They deduce it from the fact that for every separable complete metric space X there is an isometric copy $Z(X)$ of X in \mathbb{U} such that $F(\mathbb{U}) \ni X \mapsto Z(X) \in F(\mathbb{U})$ is Borel and for every $X, Y \in F(\mathbb{U})$ if X and Y are isometric, then there is an isometry $\varphi \in \text{Iso}(\mathbb{U})$ such that $\varphi''Z(X) = Z(Y)$. This follows from the fact (see Gromov [36, Page 79] and [34, Lemma 2.2]) that for every separable metric space there is a metric extension X^* of X (obtained via the Katětov construction) that is canonically isometric to the Urysohn space \mathbb{U} and such that any isometry $\varphi : X \rightarrow Y$ extends to an isometry $\varphi^* : X^* \rightarrow Y^*$. For more details on this construction see [33, Page 325]. Exactly the same arguments apply to metric spaces of diameter bounded by 1 and the Urysohn sphere \mathbb{U}_1 , which gives the following.

Proposition 3.2 *There is a Borel map $Z : F_p(\mathbb{U}_1) \rightarrow F_p(\mathbb{U}_1)$ such that for every $X \in F_p(\mathbb{U}_1)$ there exists an isometry $\theta_X : X \rightarrow Z(X)$ and if $X, Y \in F_p(\mathbb{U}_1)$ are isometric via an isometry $\varphi : X \rightarrow Y$, then there is an isometry $\tilde{\varphi} \in \text{Iso}(\mathbb{U}_1)$ with $\tilde{\varphi}''Z(X) = Z(Y)$ such that $\tilde{\varphi}$ extends $\theta_Y\varphi(\theta_X)^{-1}$.*

4 Approximate intertwining

Say that an inverse system $(S_n, \pi_n : n \in \mathbb{N})$ of simplices is *increasing* if $S_n \subseteq S_{n+1}$ is a face of S_{n+1} for each $n \in \mathbb{N}$ and $\pi_n(s) = s$ for each $s \in S_n$. Note that if $(S_n : n \in \mathbb{N})$ is an increasing system of simplices, then also $S_i \subseteq \varprojlim S_n$ is a face of $\varprojlim S_n$, for each $i \in \mathbb{N}$.

An *approximate intertwining* between two increasing inverse systems of simplices $(S_i : i \in \mathbb{N})$ and $(T_i : i \in \mathbb{N})$ is a sequence of affine continuous injective maps $\varphi_i : S_{n_i} \rightarrow T_{m_i}$ and $\psi_i : T_{m_i} \rightarrow S_{n_{i+1}}$ (for some increasing sequences n_i and m_i of natural numbers) such that $\varphi_n \subseteq \psi_n^{-1}$, $\psi_n \subseteq \varphi_{n+1}^{-1}$ (recall that the systems are increasing) and for each $\varepsilon > 0$ there exists i such that for each $j > i$ and for each $x \in S_{n_j}$ and $y \in T_{m_j}$ we have

- $d_{T_{m_i}}(\varphi_i(x \upharpoonright S_{n_i}), \varphi_j(x) \upharpoonright T_{m_i}) < \varepsilon$,
- $d_{S_{n_{i+1}}}(\psi_i(y \upharpoonright T_{m_i}), \psi_j(y) \upharpoonright S_{n_{i+1}}) < \varepsilon$.

Proposition 4.1 *Suppose $(S_i : i \in \mathbb{N})$ and $(T_i : i \in \mathbb{N})$ are two increasing inverse systems of simplices and there is an approximate intertwining between these systems. Then $\varprojlim S_i$ and $\varprojlim T_i$ are affinely homeomorphic via a map which extends all the maps in the approximate intertwining.*

Proof This is a back-and-forth argument. Write $S = \varprojlim S_i$ and $T = \varprojlim T_i$. Let $(\varphi_n : n \in \mathbb{N})$ and $(\psi_n : n \in \mathbb{N})$ form an approximate intertwining and for simplicity assume that $n_i = m_i = i$. For $x \in S$ and $n < k$ look at the point $\varphi_k(x \upharpoonright S_k) \upharpoonright T_n$ which is obtained by first projecting x to S_k , then applying φ_k and finally projecting to T_n . Note that for each $x \in S$ and for each $n \in \mathbb{N}$ the sequence of points $(\varphi_k(x \upharpoonright S_k) \upharpoonright T_n : k \in \mathbb{N})$ in T_n is Cauchy, by the property of approximate intertwining. For $x \in S$ write $x_n = \lim_k \varphi_k(x \upharpoonright S_k) \upharpoonright T_n$ for the limit of this sequence. Note that for each $n \in \mathbb{N}$ the point x_n belongs to T_n and given $n < m \in \mathbb{N}$ we have $x_m \upharpoonright T_n = x_n$ since $(\lim_k \varphi_k(x \upharpoonright S_k) \upharpoonright T_m) \upharpoonright T_n = \lim_k (\varphi_k(x \upharpoonright S_k) \upharpoonright T_m) \upharpoonright T_n = \lim_k \varphi_k(x \upharpoonright S_k) \upharpoonright T_n$. Thus, for each $x \in S$ there exists a point $y \in T$ whose projection to T_n is x_n , for every n . Write $\varphi(x) = y$. Note that the map φ is affine since the composition of φ with the projection to T_n is affine (as the limit of affine maps) for each $n \in \mathbb{N}$. Note also that for each $n \in \mathbb{N}$ the restriction of φ to S_n is equal to φ_n since if $x \in S_n$, then $\varphi_k(x \upharpoonright S_k) = \varphi_k(x) = \varphi_n(x)$ for each $k > n$.

Claim 4.2 *The map φ is continuous.*

Proof Fix $\varepsilon > 0$. We need $\delta > 0$ such that if $x_1, x_2 \in S$ are such that $d_S(x_1, x_2) < \delta$, then $d_T(\varphi(x_1), \varphi(x_2)) < \varepsilon$. Fix $n \in \mathbb{N}$ big enough so that T_n is $\varepsilon/3$ -dense in T and $\varphi(x)$ is $\varepsilon/3$ -close to $\varphi_n(x \upharpoonright S_n)$. Let $\delta > 0$ be such that if $d_{S_n}(x_1, x_2) < \delta$, then $d_{T_n}(\varphi_n(x_1), \varphi_n(x_2)) < \varepsilon/3$. Then, clearly, δ is as needed. \square

Analogously define the map $\psi : T \rightarrow S$ and argue that it is affine and continuous. Now, the facts that $\bigcup_n S_n$ is dense in S and $\bigcup_n T_n$ is dense in T , φ extends φ_n and ψ extends ψ_n for each $n \in \mathbb{N}$, imply that $\varphi^{-1} = \psi$, as $\varphi_n \subseteq \psi_n^{-1}$ and $\psi_n \subseteq \varphi_{n+1}^{-1}$. \square

5 S -extensions of metric spaces

In this section we define the S -extensions of metric spaces. The definitions below stem from a simple observation that any finite metric space (X, d_X) is isometric to the extreme boundary of a convex compact subset of ℓ_∞^n for some $n \in \mathbb{N}$. Indeed, if $X = \{x_1, \dots, x_n\}$, then the map $x_i \mapsto a_i = (d_X(x_i, x_1), \dots, d_X(x_i, x_n))$ maps X isometrically into the extreme boundary of the convex hull of $\{a_i : i \leq n\}$ taken with the ℓ_∞ metric.

Given a sequence C_n of compact convex subsets of a compact convex set C in a locally convex topological vector space, say that C_n is *convergent* if it convergent in $K(C)$ (i.e. in the Hausdorff metric) and write $\lim_n C_n$ for the limit. Note that any increasing sequence of compact convex sets is convergent and then $\lim_n C_n$ is equal to the closure of its union.

Given a metric space (X, d_X) , together with its (enumerated) countable dense set $D \subseteq X$ and a countable (enumerated) family $F \subseteq C(X, [0, 1])$ of continuous functions with values in the unit interval, we will form a convex compact set $S(X, d_X, D, F)$. Let $D = (d_n : n \in \mathbb{N})$ and $F = (f_n : n \in \mathbb{N})$. Consider the vectors $a_n = (f_1(d_n), f_2(d_n), \dots) \in [0, 1]^\mathbb{N}$ and let $S_n(X, d_X, D, F)$ be the convex hull of the set $\{a_i : i \leq n\}$ in $[0, 1]^\mathbb{N}$. Note that the sequence $S_n(X, d_X, D, F)$ is increasing, hence convergent in $K([0, 1]^\mathbb{N})$.

Definition 5.1 Given a metric space (X, d_X) define $S(X, d_X, D, F)$ as $\lim_{n \rightarrow \infty} S_n(X, d_X, D, F)$.

We will always consider $S(X, d_X, D, F)$ with the metric $d_{[0, 1]^\mathbb{N}}$ induced from the Hilbert cube. Note that since all $S_n(X, d_X, D, F)$'s are convex, so is $S(X, d_X, D, F)$. In general, this is all we know about this set. We will show, however, that if the functions F are carefully chosen, then $S(X, d_X, D, F)$ is a simplex and its set of extreme points contains a dense homeomorphic copy of X .

For each n write $S_n^n(X, d_X, D, F)$ for the projection of $S_n(X, d_X, D, F)$ into the first n -many coordinates. Treating $[0, 1]^n$ as a subset of the Hilbert cube (by padding sequences with zeros) note that the Hilbert cube is the limit of the sets $[0, 1]^n$. In the same way, $S_n^n(X, d_X, D, F)$ is treated as a subset of the Hilbert cube (also by padding sequences with zeros) and, under this identification, $S(X, d_X, D, F)$ is the limit of the sets $S_n^n(X, d_X, D, F)$.

We first show that the definition of $S(X, d_X, D, F)$ does not depend on the choice of the dense set D , in particular it does not depend on the enumeration

of D . From the point of view of further applications, we would need the fact that if D and E are two countable dense sets in X , then $S(X, d_X, D, F)$ and $S(X, d_X, E, F)$ are affinely homeomorphic. However, they are actually the same set.

Lemma 5.2 *If F is a countable family of Lipschitz 1 functions and D, E are two countable dense subsets of X , then $S(X, d_X, D, F)$ and $S(X, d_X, E, F)$ are equal.*

Proof This is straightforward. Using the fact that both D and E are dense in X , conclude that the Hausdorff distance of $S(X, d_X, D, F)$ and $S(X, d_X, E, F)$ is arbitrarily small. Since $S(X, d_X, D, F)$ and $S(X, d_X, E, F)$ are compact, they must be then equal. \square

In the sequel, we write $S(X, d_X, F)$ rather than $S(X, d_X, D, F)$. Another way of stating the previous lemma is then to say that $S(X, d_X, F)$ is the closed convex hull of the set $\{(f_1(x), f_2(x), \dots) : x \in X\}$. In principle, this can be taken as the definition of $S(X, d_X, F)$ but in further arguments we will use the approximation of $S(X, d_X, F)$ by $S_n(X, d_X, D, F)$.

Definition 5.3 We say that a countable family F of Lipschitz 1 functions on a metric space (X, d_X) is *saturated* if for every $x \in X$ the distance function $z \mapsto d_X(z, x)$ belongs to the uniform closure of F .

Note that if D is a dense subset of X , then the family of distance functions to the points from D is always saturated. Similarly, if X is embedded into the Urysohn space and D is a fixed dense subset of the Urysohn space, then the family of distance functions (defined on X) to the points in D is also saturated. In the main part of the proof we will use saturated families of this form, however, some facts below will be stated for arbitrary saturated families, in sake of generality.

Note that D can be seen as a subset of $S(X, d_X, D, F)$ via the map $d \mapsto (f_1(d), f_2(d), \dots)$. Note also that since all distance functions to the points of D are in the uniform closure of F , the above map is an embedding. Denote this map by i_F^D .

Lemma 5.4 *If F is saturated, then there is a canonical homeomorphic embedding i_F of X into $S(X, d_X, F)$ which extends i_F^D for all countable dense $D \subseteq X$ and is Lipschitz 1.*

Proof Write $i_F(x) = (f_1(x), f_2(x), \dots)$ and note that $i_F(x) \in S(X, d_X, F)$ for every $x \in X$ by Lemma 5.2. It is clear that, when viewed as a map to $S(X, d_X, D, F)$, i_F extends i_F^D . The fact that i_F is an embedding of X into $S(X, d_X, D, F)$ follows from the fact that all distance functions to the points in D are in the uniform closure of F .

To see that i_F is a homeomorphism, suppose that $x_n \in X$ and $x \in X$ are such that $i_F(x_n) \rightarrow i_F(x)$. This means that $f(x_n) \rightarrow f(x)$ for every $f \in F$. Again, since the distance functions from all points in X are in the uniform closure of F we have $d(x_n, y) \rightarrow d(x, y)$ for every $y \in X$ and hence $x_n \rightarrow x$, which shows that i_F is a homeomorphism.

Finally, the fact that i_F is Lipschitz 1 follows immediately from the fact that all functions in F are Lipschitz 1. \square

In the sequel we will abuse notation and treat X as a subset of $S(X, d_X, F)$, unless this can cause confusion.

Now, we need to locate the extreme points of the sets $S(X, d_X, F)$. Notice that if $F_1 \subseteq F_2$ are two families of Lipschitz 1 functions, then there is a natural affine continuous projection from $S(X, d_X, F_2)$ to $S(X, d_X, F_1)$ (which forgets the coordinates from $F_2 \setminus F_1$).

Note also that if K and L are convex compact sets and $\varphi : K \rightarrow L$ is an affine continuous surjection, then $\text{ext}(L) \subseteq \varphi''\text{ext}(K)$. This implies that the larger the family F we take, the more extreme points we get in $S(X, d_X, F)$.

Lemma 5.5 *If F is saturated, then the points of X are extreme points in $S(X, d_X, F)$.*

Given a compact convex set C in a locally convex vector topological space, with a metric d_C on C , and a subset $A \subseteq C$ we say that A is an ε -face provided that for every $x, y \in C$ if $\frac{1}{2}(x + y) \in A$, then both $d_C(x, A)$ and $d_C(y, A)$ are smaller than ε . Note that if $A \subseteq C$ is written as $A = \bigcap_n A_n$ so that each A_n is closed convex and for each $\varepsilon > 0$ there is n_0 such that A_n is an ε -face for each $n > n_0$, then A is a face.

Proof of Lemma 5.5 We will again look at $S(X, d_X, D, F)$ for a fixed countable dense set $D \subseteq X$. By Proposition 5.2, it is enough to show that the points of D are extreme in $S(X, d_X, D, F)$. Moreover, it is enough to show that given the enumeration of D as (d_1, d_2, \dots) , the point d_1 is extreme in $S(X, d_X, D, F)$.

Claim 5.6 *For any $A \subseteq X$ we have $\text{diam}_{S(X, d_X, F)}(\text{conv}(i_F A)) \leq \text{diam}_X(A)$.*

Proof This follows directly from the fact that i_F is Lipschitz 1 and from local convexity of the Hilbert cube. \square

Write A_n for the closed convex hull of the ball in X around d_1 of diameter $1/n$. Claim 5.6 implies that $\bigcap_n A_n$ contains only one point, and thus is the singleton $\{d_1\}$. We will show that for each $\varepsilon > 0$ there is n_0 such that A_n is an ε -face for every $n > n_0$.

Fix $\varepsilon > 0$ and find a function $f \in F$ which is $\varepsilon/8$ -uniformly close to the distance function $z \mapsto d(z, d_1)$. Without loss of generality assume that $f = f_1$.

Suppose now that $n > 8/\varepsilon$ and $y, z \in S(X, d_X, F)$ are such that $\frac{1}{2}(y+z) \in A_n$. Approximate y with y' and z with z' such that $d_{S(X, d_X, F)}(y, y') < \varepsilon/8$, $d_{S(X, d_X, F)}(z, z') < \varepsilon/8$ and $y', z' \in S_k(X, d_X, D, F)$ for some $k \in \mathbb{N}$. Note that then $\frac{1}{2}(y' + z')$ is $\varepsilon/8$ -close to A_n . Since the first coordinate of every point in A_n is smaller than $1/n + \varepsilon/8$, the first coordinate of $\frac{1}{2}(y' + z')$ is smaller than $1/n + \varepsilon/4$. Hence, the first coordinates of both y' and z' are smaller than $2(1/n + \varepsilon/4)$, which is smaller than $3\varepsilon/4$. Recall that we have

$$\|f_1 - d_X(\cdot, d_1)\|_\infty < \varepsilon/8. \quad (1)$$

Suppose that

$$y' = \left(\sum_{i=1}^k \alpha_i f_1(d_i), \sum_{i=1}^k \alpha_i f_2(d_i), \dots \right)$$

for some $\alpha_1, \dots, \alpha_k \geq 0$ with $\sum_{i=1}^k \alpha_i = 1$. Since $\sum_{i=1}^k \alpha_i f_1(d_i) \leq \frac{3\varepsilon}{4}$, by (1) we have $\sum_{i=1}^k \alpha_i d_X(d_i, d_1) \leq \frac{3\varepsilon}{4} + \frac{\varepsilon}{8}$. Now, for each j the function f_j is Lipschitz 1, so

$$|f_j(d_i) - f_j(d_1)| \leq d_X(d_i, d_1). \quad (2)$$

Recall that the point d_1 is identified with the sequence $(f_1(d_1), f_2(d_1), \dots)$. With this in mind, by (2) for each j we have

$$\begin{aligned} |y'(j) - d_1(j)| &= \left| \sum_{i=1}^k \alpha_i f_j(d_i) - f_j(d_1) \right| \\ &= \left| \sum_{i=1}^k \alpha_i f_j(d_i) - \sum_{i=1}^k \alpha_i f_j(d_1) \right| \\ &\leq \sum_{i=1}^k \alpha_i d_X(d_i, d_1) \leq \frac{3\varepsilon}{4} + \frac{\varepsilon}{8} \end{aligned}$$

This, and an analogous argument for z' in place of y' , shows that

$$d_{S(X, d_X, F)}(y', d_1) < \frac{3\varepsilon}{4} + \frac{\varepsilon}{8} \quad \text{as well as} \quad d_{S(X, d_X, F)}(z', d_1) < \frac{3\varepsilon}{4} + \frac{\varepsilon}{8}.$$

This implies that $d_{S(X, d_X, F)}(y, d_1) < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$ and $d_{S(X, d_X, F)}(z, d_1) < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$, which shows that A_n is an ε -face. \square

The fact that X is dense in the set of extreme points of $S(X, d_X, F)$ will follow from the following general lemma.

Lemma 5.7 *Suppose C_n is an increasing sequence of compact convex subsets of a metrizable convex compact set C in a locally convex topological vector space. Then $\text{ext}(\lim_n C_n)$ is contained in the closure of $\bigcup_n \text{ext}(C_n)$.*

Proof Write K for the closure of $\bigcup_n \text{ext}(C_n)$ and suppose $x \in \lim_n C_n$ is such that $x \notin K$. We need to show that x is not an extreme point of $\lim_n C_n$. Note that [44, Chapter 15, Proposition 2.3] x is a barycenter of a probability measure μ concentrated on K . By a theorem of Bauer [7] (see also [62, Proposition 1.4]) if x is an extreme point and a barycenter of a probability measure μ , then $\mu = \delta_x$. But μ is concentrated on K , so cannot be equal to δ_x . \square

This immediately gives the following.

Corollary 5.8 *If F is saturated, then X is dense in the set of extreme points of $S(X, d_X, F)$.*

Let us now see some examples of convex compact sets that can arise as $S(X, d_X, F)$. The next proposition will not be used later in the proof but it shows some ideas behind the constructions that follow.

In the following proposition, for a compact metric space X , write $P(X)$ for the Bauer simplex of all Borel probability measures on X . The extreme boundary of $P(X)$ is canonically homeomorphic to X and any simplex whose extreme boundary is homeomorphic to X is canonically affinely homeomorphic to $P(X)$ (this follows for example from the positive solution to the Dirichlet extension problem for Bauer simplices [4]).

As a comment to the assumption of the following proposition, note that if (X, d_X) is compact, then Lipschitz functions are dense in $C(X)$, by the Stone–Weierstrass theorem. Thus, if (X, d_X) is compact, then the set of all Lipschitz 1 functions is linearly dense in $C(X)$.

Proposition 5.9 *Suppose (X, d_X) is compact, F is saturated and linearly dense in $C(X)$. Then $S(X, d_X, F)$ is affinely homeomorphic to the Bauer simplex $P(X)$.*

Proof Note that the set of extreme points of $S(X, d_X, F)$ is equal to X by Lemma 5.5 and Corollary 5.8. We need to prove that $S(X, d_X, F)$ is a simplex. For that, suppose μ, ν are two distinct probability measures on X . By linear density of F in $C(X)$, there is $f \in F$ such that $\int f d\mu \neq \int f d\nu$. Without loss of generality, assume that $f = f_1$. Pick a countable dense set $D \subseteq X$ and choose two sequences of atomic measures μ_n and ν_n concentrated on D such that $\mu_n \rightarrow \mu$ and $\nu_n \rightarrow \nu$. Note that each μ_n and ν_n has a barycenter in one of the sets $S_k(X, d_X, D, F)$, for if

$$\mu_n = \sum_{i=1}^{k_n} \alpha_i^n \delta_{d_i^n} \quad \text{with} \quad d_i^n \in D, \alpha_i^n \geq 0, \sum_{i=1}^{k_n} \alpha_i^n = 1,$$

$$\nu_n = \sum_{i=1}^{l_n} \beta_i^n \delta_{e_i^n} \quad \text{with} \quad e_i^n \in D, \beta_i^n \geq 0, \sum_{i=1}^{l_n} \beta_i^n = 1,$$

then the barycenter of μ_n is $\sum_{i=1}^{k_n} \alpha_i^n d_i^n$ and the barycenter of ν_n is $\sum_{i=1}^{l_n} \beta_i^n e_i^n$. Note that then $\sum_{i=1}^{k_n} \alpha_i^n f_1(d_i^n)$ and $\sum_{i=1}^{l_n} \beta_i^n f_1(e_i^n)$ are the first coordinates in $[0, 1]^{\mathbb{N}}$ of the barycenters of μ_n and ν_n . Thus, the first coordinate of the barycenter of μ is the limit of the first coordinates of the barycenters of μ_n [44, Chapter 15, Proposition 2.2] and is equal to

$$\lim_n \sum_{i=1}^{k_n} \alpha_i^n f_1(d_i^n) = \lim_n \int f_1 d\mu_n = \int f_1 d\mu$$

and the first coordinate of the barycenter of ν is the limit of the first coordinates of the barycenters of ν_n and is equal to

$$\lim_n \sum_{i=1}^{l_n} \beta_i^n f_1(e_i^n) = \lim_n \int f_1 d\nu_n = \int f_1 d\nu.$$

Thus, μ and ν have different barycenters, which shows that $S(X, d_X, F)$ is a simplex. \square

Now we need to see how $S(X, d_X, F)$ depends on the choice of F . First note that it does not depend on the enumeration of F .

Lemma 5.10 *Suppose F and G enumerates the same set of functions as F . Then $S(X, d_X, F)$ and $S(X, d_X, G)$ are affinely homeomorphic.*

Proof Let $\pi : \mathbb{N} \rightarrow \mathbb{N}$ be a permutation such that $G = (f_{\pi(1)}, f_{\pi(2)}, \dots)$. Let $h : [0, 1]^{\mathbb{N}} \rightarrow [0, 1]^{\mathbb{N}}$ be defined as $h(x_1, x_2, \dots) = (x_{\pi(1)}, x_{\pi(2)}, \dots)$ and note that h is an affine homeomorphism of the Hilbert cube which maps $S(X, d_X, F)$ to $S(X, d_X, G)$. \square

Suppose now that (X, d_X) is a metric space, $D \subseteq X$ is a countable dense set and that F consists of distance functions from the points in D . In such a case, we denote F by $d_X D$. Note that $d_X D$ always consists of Lipschitz 1 functions and its uniform closure contains all distance functions to the points in X .

Proposition 5.11 *Suppose (X, d_X) is separable, complete and $D, E \subseteq X$ are two countable dense sets. Then there is an affine homeomorphism $\tau_D^E :$*

$S(X, d_X, d_X D) \rightarrow S(X, d_X, d_X E)$ such that the following diagram commutes.

$$\begin{array}{ccc} S(X, d_X, d_X D) & \xrightarrow{\tau_D^E} & S(X, d_X, d_X E) \\ & \nwarrow i_{d_X D} \quad \nearrow i_{d_X E} & \\ & X & \end{array}$$

The map τ_D^E should be treated as the transition map between the coordinate systems of D and E .

Proof First note that isolated points of X must belong to both D and E and hence if X_1 is the set of isolated points, then by Lemma 5.10 we can assume that X_1 is enumerated in the same way in D and E . Thus, without loss of generality we can assume that X has no isolated points. Moreover, we can assume that D and E are disjoint since we can always use a third countable dense set which is disjoint from both D and E . Write $D = (d_0, d_1, \dots)$ and let $D' = (d_1, d_2, \dots)$.

Claim 5.12 $S(X, d_X, d_X D)$ and $S(X, d_X, d_X D')$ are affinely homeomorphic via a map $\tau_D^{D'} : S(X, d_X, d_X D) \rightarrow S(X, d_X, d_X D')$ such that $\tau_D^{D'} \circ i_{d_X D} = i_{d_X D'}$.

Proof Write $C_n = S_n(X, d_X, D, d_X D)$ and $B_n = S_n(X, d_X, D, d_X D')$ for each $n \in \mathbb{N}$. Write also $[0, 1]^{\mathbb{N}}$ as $[0, 1]^{\{2,3,\dots\}} \times [0, 1]$ interpreting the second coordinate of the product as the first coordinate in the Hilbert cube. Pick a subsequence $d_{k_n} \rightarrow d_0$ in D and note that the distance function to d_0 is the uniform limit of the distance functions to d_{k_n} 's. By Proposition 2.1, this implies that on $\bigcup_n C_n$ the first coordinate is the uniform limit of the coordinates numbered with k_n 's. Thus, $\bigcup_n C_n$ treated as a subset of $[0, 1]^{\{2,3,\dots\}} \times [0, 1]$, is a graph of an affine function, say a , defined on $\bigcup_n B_n$. Note that since a is the uniform limit of the sequence of functions given by coordinates in $\{k_1, k_2, \dots\}$, it follows that this sequence of functions is uniformly Cauchy on $\bigcup_n B_n$. This sequence is also uniformly Cauchy on the closure of $\bigcup_n B_n$, and hence uniformly convergent. Thus, the function a uniquely extends to an affine function defined on $\overline{\bigcup_n B_n} = S(X, d_X, d_X D')$. Note that the graph of this unique extension is equal to the closure of $\bigcup_n C_n$, i.e. $S(X, d_X, d_X D)$. Given that, the projection function from $[0, 1]^{\{2,3,\dots\}} \times [0, 1]$ to $[0, 1]^{\{2,3,\dots\}}$ is an affine isomorphism from $S(X, d_X, d_X D)$ to $S(X, d_X, d_X D')$. Write $\tau_D^{D'}$ for this isomorphism and note that since it just erases the first coordinate, we have $\tau_D^{D'} \circ i_{d_X D} = i_{d_X D'}$. \square

Write now D_n for the sequence $(e_1, e_2, \dots, e_n, d_{n+1}, d_{n+1}, \dots)$.

Claim 5.13 For each $n \in \mathbb{N}$, there is an affine homeomorphism $\tau_n : S(X, d_X, d_X D) \rightarrow S(X, d_X, d_X D_n)$ such that $\tau_n \circ i_{d_X D} = i_{d_X D_n}$.

Proof It is enough to show that there is an affine homeomorphism $\tau'_n : S(X, d_X, d_X D_n) \rightarrow S(X, d_X, d_X D_{n+1})$ with $\tau'_n \circ i_{D_n} = i_{D_{n+1}}$ and without loss of generality assume that $n = 0$. But this follows from Claim 5.12 since both $S(X, d_X, d_X D)$ and $S(X, d_X, d_X D_1)$ are affinely homeomorphic to $S(X, d_X, d_X D')$ via maps which make the appropriate diagrams commute. \square

Now note that since $\tau_{n+1}\tau_n^{-1}$ changes only the n th coordinate, the sequence τ_n is uniformly Cauchy with respect to the Hilbert cube metric, and thus converges to an affine map $\tau_D^E : S(X, d_X, d_X D) \rightarrow [0, 1]^{\mathbb{N}}$. Moreover, since the distance of $\text{rng}\tau_n$ to $S_n(X, d_X, D, d_X E)$ is smaller than 2^{-n} , we have that τ_D^E is an affine map from $S(X, d_X, d_X D)$ to $S(X, d_X, d_X E)$. In the same way construct maps $\tau^n : S(X, d_X, d_X E) \rightarrow S(X, d_X, d_X E_n)$ with $E_n = (d_1, d_2, \dots, d_n, e_{n+1}, e_{n+2}, \dots)$ and note that their limit is an affine function $\tau_E^D : S(X, d_X, d_X E) \rightarrow S(X, d_X, d_X D)$. The maps are clearly inverse to each other, so τ_D^E is an affine homeomorphism. Finally, $\tau_D^E \circ i_{d_X D} = i_{d_X E}$ follows from the fact that $\tau_n \circ i_{d_X D} = i_{d_X D_n}$ holds for each n and $i_{d_X D_n} \rightarrow i_{d_X E}$. This ends the proof. \square

The following definition and proposition will not be used in the proof but we state it here since they may be useful in answering some questions posed in Sect. 9. Note that if we want the set $S(X, d_X, F)$ to be a simplex, a natural condition for the family F is the following.

Definition 5.14 A countable family F of Lipschitz 1 functions is called *independent* if for every $x_1, \dots, x_n \in X$ there are $f_1, \dots, f_n \in F$ such that the vectors $(f_1(x_1), \dots, f_n(x_1)), \dots, (f_1(x_n), \dots, f_n(x_n)) \in \mathbb{R}^n$ are linearly independent.

The following proposition is stated for the Urysohn sphere \mathbb{U}_1 but it also holds true (with the same proof) for the Urysohn space \mathbb{U} . We say that a function $f : X \rightarrow [0, \infty)$ defined on a metric space (X, d_X) is a *Katětov function* if for every $x, y \in X$ we have $|f(x) - f(y)| \leq d_X(x, y) \leq f(x) + f(y)$ (cf. [61, Lemma 5.1.22] and [33, Definition 1.2.1]).

Proposition 5.15 For any countable dense subset D of the Urysohn sphere \mathbb{U}_1 , the set $d_{\mathbb{U}_1} D$ is independent.

Proof Write d for the metric $d_{\mathbb{U}_1}$ on \mathbb{U}_1 . Pick a countable dense set $D \subseteq \mathbb{U}_1$ and let $x_1, \dots, x_n \in \mathbb{U}_1$. We show that there are $d_1, \dots, d_n \in D$ such that the matrix

$$\begin{pmatrix} d(x_1, d_1) & \dots & d(x_1, d_n) \\ \vdots & \ddots & \vdots \\ d(x_n, d_1) & \dots & d(x_n, d_n) \end{pmatrix}$$

is invertible. The proof is by induction. For $n = 1$, any $d_1 \neq x_1$ will do. Suppose $n > 1$ and $x_1, \dots, x_n \in \mathbb{U}$ are given. Pick any d_1, \dots, d_{n-1} that witness the inductive assumption for x_1, \dots, x_{n-1} and let $d' \in D$ be such that $d(x_n, d') < d(x_i, d')$ for all $i \neq n$. Consider the function

$$\varepsilon \mapsto \det \begin{pmatrix} d(x_1, d_1) & \dots & d(x_1, d_{n-1}) & d(x_1, d') \\ \vdots & \ddots & \vdots & \vdots \\ d(x_n, d_1) & \dots & d(x_n, d_{n-1}) & d(x_n, d') + \varepsilon \end{pmatrix}$$

and note that it is a nonzero linear function, so there are arbitrarily small $\varepsilon > 0$ at which it does not vanish. Pick such $\varepsilon_0 > 0$ which is smaller than $\min\{d(x_i, d') - d(x_n, d') : i < n\}$. Note now that the function $f : \{x_1, \dots, x_n\} \rightarrow [0, 1]$ given by $f(x_i) = d(x_i, d')$ if $i < n$ and $f(x_n) = d(x_n, d') + \varepsilon_0$ is a Katětov function with values in $[0, 1]$. Thus, there is $y \in \mathbb{U}_1$ which realizes f . This means that

$$\det \begin{pmatrix} d(x_1, d_1) & \dots & d(x_1, d_{n-1}) & d(x_1, y) \\ \vdots & \ddots & \vdots & \vdots \\ d(x_n, d_1) & \dots & d(x_n, d_{n-1}) & d(x_n, y) \end{pmatrix} \neq 0$$

Since the set of such y 's is clearly open, we can find one, say d_n , in D . \square

Now we come to the question how to make sure that our convex compact sets are simplices. The Urysohn space has the extension property saying that every Katětov function defined on its finite subset is realized as a distance function to some point in the space. Huhunaišvili [42] (cf also [9, 36, 45, 59, 60]) showed that the same is true for Katětov functions defined on compact subsets of the Urysohn space. This is probably the strongest result that can guarantee that certain Katětov functions can be realized in the Urysohn space (or the Urysohn sphere). The following construction is motivated by the need of realizing Katětov functions defined on non-compact subspaces of the Urysohn sphere.

Let (X, d_X) be a separable metric space of diameter bounded by 1 and let $D \subseteq X$ be its dense countable subset. Let $R(D)$ be the ring of functions generated by the distance functions $x \mapsto d_X(x, d)$ for $d \in D$ and all rational constant functions. Note that $R(D)$ is countable and all functions in $R(D)$ are Lipschitz. Let $R_1(D)$ be the family of functions in $R(D)$ which are Lipschitz 1 and have the range contained in $[\frac{1}{2}, 1]$. Note that dividing a bounded Lipschitz function by an appropriately large constant, we get a Lipschitz 1 function with the range contained in $[-\frac{1}{4}, \frac{1}{4}]$. Next, adding $\frac{3}{4}$ we get a Lipschitz 1 function with $\text{rng}(f) \subseteq [\frac{1}{2}, 1]$. This shows that $R_1(D)$ is linearly dense in $R(D)$. On the other hand, since d_X is bounded by 1, every function in $R_1(D)$ is a Katětov

function. Recall that the elements of X are identified with Katětov functions on X by the Kuratowski construction [33, Chapter 1.2], i.e. $x \in X$ is identified with the function $y \mapsto d_X(x, y)$. Write $F(X)$ for the family of all finitely supported (see [33, Definition 1.2.2]) Katětov functions on X with values in $[0, 1]$. Let $E(X, d_X, D)$ be the completion of the space $F(X) \cup R_1(D)$ with the sup metric. Note that $E(X, d_X, D)$ is an extension of X (as $F(X)$ contains all functions $y \mapsto d_X(x, y)$ for $x \in X$), is separable and realizes all finitely supported Katětov functions on X . However, it is slightly bigger than the usual one-step Katětov extension since we also have realized the functions in $R_1(D)$. A standard argument shows that if $\varphi : X \rightarrow X$ is an isometry, then φ extends to an isometry φ' of $E(X, d_X, D)$ and the definition of $E(X, d_X, D)$ does not depend on the choice of the dense set D . Thus, slightly abusing notation, we write $E(X, d_X)$ for $E(X, d_X, D)$. Note, however, that given D , we have a canonical countable dense set D^E in $E(X, d_X, D)$ consisting of $R_1(D)$ and all Katětov functions finitely supported on a subset of D and assuming rational values on their support.

Now, similarly as in the Katětov construction of the Urysohn space, we iterate the above extension construction infinitely many times.

Definition 5.16 Given a separable metric space (X, d_X) and its countable dense subset $D \subseteq X$ define inductively $E^n(X, d_X, D)$ and $D^n(X, d_X, D) \subseteq E^n(X, d_X, D)$ as follows. $E^0(X, d_X, D) = (X, d_X)$ and $D_0(X, d_X, D) = D$. Given $E^n(X, d_X, D)$ and $D^n(X, d_X, D)$, which is a countable dense subset of $E^n(X, d_X, D)$ let $E^{n+1}(X, d_X, D) = E(E^n(X, d_X, D), D^n(X, d_X, D))$ and let $D^{n+1}(X, d_X, D)$ be $D^n(X, d_X, D)^E$ (cf the remarks proceeding this definition). Write $E^\infty(X, d_X, D)$ for the completion of the space $\bigcup E^n(X, d_X, D)$ and $D^\infty(X, d_X, D)$ for $\bigcup_n D^n(X, d_X, D)$.

Again, the construction of $E^\infty(X, d_X, D)$ does not depend on the initial choice of the countable dense set D and we will abuse notation writing $E^\infty(X, d_X)$ for $E^\infty(X, d_X, D)$, unless this can cause confusion. Note that the space $E^\infty(X, d_X)$ is actually isometric to the Urysohn sphere since it realizes all finitely supported Katětov functions with values in $[0, 1]$. Note also that if $\varphi : X \rightarrow X$ is an isometry, then φ extends canonically to an isometry $\varphi^\infty : E^\infty(X, d_X) \rightarrow E^\infty(X, d_X)$. This follows in the same way as the analogous extension property is proved for the Katětov extensions [61, Page 115] from the (above mentioned) fact that φ extends to $E(X, d_X)$.

Given a metric space (X, d_X) and its countable dense subset D , write $d_{E^\infty(X, d_X)} D^\infty(X, d_X)$ for the family of functions on X of the form $x \mapsto d_{E^\infty(X, d_X)}(x, d)$ for $d \in D^\infty(X, d_X, D)$. Write also $R^\infty(X, d_X, D)$ for the ring of functions on X generated by $d_{E^\infty(X, d_X)} D^\infty(X, d_X)$. The next proposition follows rather immediately from the construction.

Proposition 5.17 *Let (X, d_X) be a separable metric space of diameter bounded by 1 and let $D \subseteq X$ be a countable dense set. Then $R^\infty(X, d_X, D)$ is contained in the linear span of $d_{E^\infty(X, d_X)} D^\infty(X, d_X)$.*

Proof Write D^∞ for $D^\infty(X, d_X, d)$ and dD^∞ for $d_{E^\infty(X, d_X)} D^\infty(X, d_X)$. Every function f in $R^\infty(X, d_X, D)$ is a polynomial in finitely many functions in dD^∞ , which in turn are the distance functions to finitely many points in D^∞ . Thus, there is $n \in \mathbb{N}$ such that f belongs to the ring generated by the distance functions to $D^n(X, d_X, D)$. Then, f belongs to the linear span of distance functions to the points in $D^{n+1}(X, d_X, D)$. \square

Now, fix a countable dense set $D \subseteq \mathbb{U}_1$ and write D^∞ for $D^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}, D)$ and d_E for the metric on $E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1})$. Consider the convex compact set $S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), d_E D^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}, D))$ and write $S(\mathbb{U}_1, d_{\mathbb{U}_1})$ for the closed convex hull of \mathbb{U}_1 in $S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), d_E D^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}, D))$. Note that $S(\mathbb{U}_1, d_{\mathbb{U}_1})$ is equal to $S(\mathbb{U}_1, d_{\mathbb{U}_1}, d_E D^\infty)$. Note also that by the remarks following Definition 5.3 and the fact that $E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1})$ is isometric to the Urysohn sphere, the family $d_E D^\infty$ is saturated.

Recall (Proposition 3.2) that given a subspace (X, d_X) of \mathbb{U}_1 we write $Z(X, d_X)$ for an isometric copy of (X, d_X) appropriately embedded into \mathbb{U}_1 .

Definition 5.18 For a subspace (X, d_X) of the Urysohn sphere $(\mathbb{U}_1, d_{\mathbb{U}_1})$, write $S_Z(X, d_X)$ for $S(Z(X, d_X), d_E D^\infty)$.

Note that by Proposition 5.11 and the remarks proceeding Proposition 5.17, up to affine homeomorphism, the above definition does not depend on the choice of the dense countable set $D \subseteq \mathbb{U}_1$.

Similarly as with Proposition 5.15, the following result is stated for the Urysohn sphere but holds true for the Urysohn space as well.

Proposition 5.19 *The convex compact set $S(\mathbb{U}_1, d_{\mathbb{U}_1})$ is a simplex.*

Proof Write S for $S(\mathbb{U}_1, d_{\mathbb{U}_1})$ and d_S for the metric on S (induced from the Hilbert cube). Recall that we have fixed a countable dense set $D \subseteq \mathbb{U}_1$ and write d_E for the metric on $E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}, D)$.

Suppose μ and ν are two distinct Borel probability measures in $P(S)$ which are supported on $\text{ext}(S)$. Since as $\mathbb{U}_1 \subseteq \text{ext}(S)$ is dense by Corollary 5.8, we can pick two sequences of Borel probability measures $\mu_n \rightarrow \mu$ and $\nu_n \rightarrow \nu$ such that μ_n and ν_n are finitely supported on \mathbb{U}_1 .

Since μ and ν are distinct as elements of $P(S)$, and the coordinate functions (from $[0, 1]^\mathbb{N}$) separate points in S , by the Stone–Weierstrass theorem, there is a function $f \in C(S)$ which belongs to the ring generated by the coordinate functions and is such that $\int f d\mu \neq \int f d\nu$. Note that the restrictions of the coordinate functions to \mathbb{U}_1 are equal to the d_E -distance functions to the points in D^∞ . By Proposition 5.17 and the fact that \mathbb{U}_1 is dense in $\text{ext}(S)$, we can

assume that the restriction of f to \mathbb{U}_1 is actually equal to the d_E -distance function to, say, $z \in D^\infty$. Thus, we assume that the function f on $\text{ext}(S)$ is just one of the coordinate functions. Say it is the k th coordinate, i.e. z is the k th element of D^∞ .

Write x for the barycenter of μ and y for the barycenter of ν and let x_n be the barycenter of μ_n and y_n of ν_n . Note that $x_n \rightarrow x$ and $y_n \rightarrow y$ [44, Chapter 15, Proposition 2.2]. Now, since μ_n and ν_n are supported on \mathbb{U}_1 , similarly as in Proposition 5.9, we get that $x_n(k) = \int f d\mu_n$ and $y_n(k) = \int f d\nu_n$. But $\int f d\mu_n \rightarrow \int f d\mu$ and $\int f d\nu_n \rightarrow \int f d\nu$. Thus, $x(k) = \int f d\mu$ and $y(k) = \int f d\nu$ and hence x and y are distinct, as needed. This ends the proof. \square

Corollary 5.20 *For every closed subspace (X, d_X) of \mathbb{U}_1 the set $S_Z(X, d_X)$ is a simplex and $Z(X, d_X)$ is a dense subset of $\text{ext}(S_Z(X, d_X))$.*

Proof For simplicity identify (X, d_X) with $Z(X, d_X)$. Note that $S_Z(X, d_X)$ is equal to the closed convex hull of X in $S(\mathbb{U}_1, d_{\mathbb{U}_1})$ and X is contained in $\text{ext}(S(\mathbb{U}_1, d_{\mathbb{U}_1}))$. Thus, $S_Z(X, d_X)$ is a face of $S(\mathbb{U}_1, d_{\mathbb{U}_1})$. By Proposition 5.19, this implies [44, Chapter 15, Corollary 3.3] that $S_Z(X, d_X)$ is a simplex. Clearly, X is contained in $\text{ext}(S_Z(X, d_X))$. The fact that X is dense in $\text{ext}(S_Z(X, d_X))$ follows directly from Corollary 5.8. \square

Finally, we need to see that S -extensions of metric spaces are invariant under the isometry of the metric spaces.

Proposition 5.21 *Suppose (X, d_X) and (Y, d_Y) subspaces of \mathbb{U}_1 . If (X, d_X) and (Y, d_Y) are isometric, then the simplices $S_Z(X, d_X)$ and $S_Z(Y, d_Y)$ are affinely homeomorphic via a map that extends the isometry of $Z(X, d_X)$ and $Z(Y, d_Y)$.*

Proof For simplicity again identify (X, d_X) with $Z(X, d_X)$ and (Y, d_Y) with $Z(Y, d_Y)$. Let $\varphi : \mathbb{U}_1 \rightarrow \mathbb{U}_1$ be an isometry such that $\varphi''X = Y$. Let $D \subseteq \mathbb{U}_1$ be the fixed countable dense set and let $E = \varphi''D$. Write $\varphi^\infty : E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}) \rightarrow E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1})$ for the extension of φ and note that $(\varphi^\infty)''D^\infty = E^\infty$. Write d for the metric on $E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1})$. Note that since φ^∞ is an isometry, the sets $S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), D^\infty, dD^\infty)$ and $S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), E^\infty, dE^\infty)$ are equal and we get the following diagram:

$$\begin{array}{ccc} S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), D^\infty, dD^\infty) & \xrightarrow{\text{id}} & S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), E^\infty, dE^\infty) \\ \uparrow i_{dD^\infty} & & \uparrow i_{dE^\infty} \\ \mathbb{U}_1 & \xrightarrow{\varphi} & \mathbb{U}_1 \end{array}$$

Composing this with the diagram in Proposition 5.11 (where we take the restrictions of the maps i_{dD^∞} and i_{dE^∞} to \mathbb{U}_1), we get

$$\begin{array}{ccc}
 S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), D^\infty, dD^\infty) & \xrightarrow{\tau_{D^\infty}^{E^\infty}} & S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), D^\infty, dD^\infty) \\
 \uparrow i_{dD^\infty} & & \uparrow i_{dD^\infty} \\
 \mathbb{U}_1 & \xrightarrow{\varphi} & \mathbb{U}_1
 \end{array}$$

and since all the maps are affine and continuous, this immediately implies that $\tau_{D^\infty}^{E^\infty}$ maps $\text{conv}(i_{dD^\infty} X) = S_Z(X, d_X)$ to $\text{conv}(i_{dD^\infty} Y) = S_Z(Y, d_Y)$. \square

The simplex $S(\mathbb{U}_1, d_{\mathbb{U}_1})$ plays now the role of a universal homogeneous Choquet simplex. However, it does not seem to be affinely homeomorphic to the Poulsen simplex. On the other hand, that there is another convex compact set, which seems to be related to the Poulsen simplex. It is the set $S(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$. Write $S'(\mathbb{U}_1, d_{\mathbb{U}_1})$ for $S(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ (for a countable dense set $D \subseteq \mathbb{U}_1$). Again, up to affine homeomorphism, it does not depend on the choice of the countable set D . Now, the map from $S(\mathbb{U}_1, d_{\mathbb{U}_1})$ to $S'(\mathbb{U}_1, d_{\mathbb{U}_1})$, which forgets about the coordinates corresponding to $D^\infty \setminus D$ is clearly affine and continuous and hence $S(\mathbb{U}_1, d_{\mathbb{U}_1})$ can be treated as an unfolded version of $S'(\mathbb{U}_1, d_{\mathbb{U}_1})$. We do not know if $S'(\mathbb{U}_1, d_{\mathbb{U}_1})$ is a simplex but it seems to be more closely related to the Poulsen simplex than $S(\mathbb{U}_1, d_{\mathbb{U}_1})$.

Proposition 5.22 $S'(\mathbb{U}_1, d_{\mathbb{U}_1})$ has a dense set of extreme points.

The proof follows from the following simple claim.

Claim 5.23 If g and h are Katětov functions and $\alpha \in [0, 1]$, then $\alpha f + (1 - \alpha)g$ is also a Katětov function.

Proof This is an elementary computation. The Katětov inequalities for $\alpha f + (1 - \alpha)g$ follow directly from the Katětov inequalities for f and g . \square

Proof of Proposition 5.22 By the fact that $\mathbb{U}_1 \subseteq \text{ext}(S'(\mathbb{U}_1, d_{\mathbb{U}_1}))$, it is enough to check that \mathbb{U}_1 is dense in $S'(\mathbb{U}_1, d_{\mathbb{U}_1})$. Let $s \in S'(\mathbb{U}_1, d_{\mathbb{U}_1})$ and $\varepsilon > 0$. Find n such that $2^{-(n+1)} < \varepsilon/2$ and s is ε -close to $S_n^n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$. Write $D = (d_1, d_2, \dots)$ and $\mathbb{U}_1 \upharpoonright n$ for the image of $\mathbb{U}_1 \subseteq S(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ under the projection map from $[0, 1]^{\mathbb{N}}$ to $[0, 1]^n$. Note that it suffices to show that $S_n^n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ is contained in $\mathbb{U}_1 \upharpoonright n$. Clearly, the vertices of the simplex $S_n^n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ belong to $\mathbb{U}_1 \upharpoonright n$. For each $i \leq n$ let $f_i : \{d_1, \dots, d_n\} \rightarrow [0, 1]$ be defined $f_i(z) = d(\alpha_i, z)$. Now, Claim 5.23 implies that for every $\alpha_1, \dots, \alpha_n \in [0, 1]$ with $\sum_i \alpha_i = 1$ the function $\sum_i \alpha_i f_i$ is a Katětov function with values in $[0, 1]$, thus realized in \mathbb{U}_1 . This implies that the set of points $(d(x, d_1), \dots, d(x, d_n)) \in [0, 1]^n$ for $x \in \mathbb{U}_1$ is convex. But the latter set is equal to $\mathbb{U}_1 \upharpoonright n$. Thus, since the vertices of $S_n^n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ belong to $\mathbb{U}_1 \upharpoonright n$ and $\mathbb{U}_1 \upharpoonright n$ is convex, we have that $S_n^n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ is contained in $\mathbb{U}_1 \upharpoonright n$, as needed. This ends the proof. \square

6 Iterated cone construction

Given a simplex S and a point $s \in S$ we define the *cone of S over s* as follows. Consider $Y = S \times [0, 1]$ and let

$$\text{cone}(S, s) = \text{conv}((S \times \{0\}) \cup (s, 1)),$$

where the convex hull is taken in Y (recall our convention on metrics in Sect. 2 and note that if S is a subset of the Hilbert cube, then the cone is embedded in the Hilbert cube as well).

The point $(s, 1)$ is called the *cone point* of the cone and we will identify S with $S \times \{0\} \subseteq \text{cone}(S, s)$. We also denote the cone point as $c(s)$. The cone admits a natural affine continuous map $\pi : \text{cone}(S, s) \rightarrow S$ which is just the projection map in Y . It maps the cone point $c(s)$ to the point s .

We state now a couple of basic facts on the structure of cones.

Lemma 6.1 *If S is a simplex and $s \in S$, then $\text{cone}(S, s)$ is a simplex, S is a face of $\text{cone}(S, s)$ and $\text{ext}(\text{cone}(S, s)) = \text{ext}(S) \cup \{c(s)\}$.*

Proof Note that if two points $z_1, z_2 \in S \times [0, 1]$ have an affine combination that lies in S , then both z_1 and z_2 must belong to S . This implies that S is a face of the convex compact set $\text{cone}(S, s)$ and that $\text{ext}(S) \subseteq \text{ext}(\text{cone}(S, s))$. It is clear that $c(s)$ is an extreme point of $\text{cone}(S, s)$, so indeed $\text{ext}(\text{cone}(S, s)) = \text{ext}(S) \cup \{c(s)\}$. To see that $\text{cone}(S, s)$ is a simplex note that if μ is a measure concentrated on $\text{ext}(S) \cup \{c(s)\}$ such that $\int f d\mu = 0$ for every affine continuous function on $\text{cone}(S, s)$, then $\mu(\{c(s)\}) = 0$ and hence μ must be concentrated on $\text{ext}(S)$. \square

Lemma 6.2 *Given a simplex S and two points $s_1, s_2 \in S$ we have*

$$\text{cone}(\text{cone}(S, s_1), s_2) \simeq \text{cone}(\text{cone}(S, s_2), s_1).$$

Proof Both simplices are affinely homeomorphic to the closed convex hull of $S \times \{(0, 0)\}, (s_1, 0, 1)$ and $(s_2, 1, 0)$ in $S \times [0, 1]^2$. To see the affine homeomorphism of $\text{cone}(\text{cone}(S, s_1), s_2)$ and $\text{cone}(\text{cone}(S, s_2), s_1)$ directly, write $c(s_1)$ for the cone point of $\text{cone}(S, s_1)$, $c'(s_2)$ for the cone point of $\text{cone}(\text{cone}(S, s_1), s_2)$, $c(s_2)$ for the cone point of $\text{cone}(S, s_2)$ and $c'(s_1)$ for the cone point of $\text{cone}(\text{cone}(S, s_2), s_1)$. The affine homeomorphism then maps $(1 - \beta)((1 - \alpha)x + \alpha c(s_1)) + \beta c'(s_2)$ to $(1 - \gamma)((1 - \delta)x + \delta c(s_2)) + \gamma c'(s_1)$, where $\gamma = (1 - \beta)\alpha$ and $\delta = \beta/(1 - \alpha(1 - \beta))$. \square

Given the above lemma, we use the notation $\text{cone}(S, s_1, \dots, s_n)$ to denote an iterated cone of the form $\text{cone}(\dots \text{cone}(S, s_1) \dots, s_n)$ (or with any other permutation). We also call a cone of the form $\text{cone}(S, s_1, s_2)$ a *double cone*.

Given a subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, the simplex $\text{cone}(S, s_{i_1}, \dots, s_{i_k})$ is a subset of $\text{cone}(S, s_1, \dots, s_n)$ in the same way S is a subset of $\text{cone}(S, s)$. The simplex $\text{cone}(S, s_{i_1}, \dots, s_{i_k})$ is then a face of $\text{cone}(S, s_1, \dots, s_n)$. We call a face of the form $\text{cone}(S, s_{i_1}, \dots, s_{i_k})$ in $\text{cone}(S, s_1, \dots, s_n)$ a *subcone*.

Lemma 6.3 *Given a simplex S and $s_1, \dots, s_n \in S$, if $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and $\{j_1, \dots, j_{n-k}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$, then*

$$\text{cone}(S, s_1, \dots, s_n) \simeq \text{cone}(\text{cone}(S, s_{i_1}, \dots, s_{i_k}), s_{j_1}, \dots, s_{j_{n-k}}).$$

Proof This follows directly from the definitions and Lemma 6.2. \square

Definition 6.4 Given a simplex S and its countable (enumerated) subset $D = (d_n : n \in \mathbb{N})$, define the *iterated cone over D* as follows. Let $S_0 = S$ and for each $n \in \mathbb{N}$ let $S_n = \text{cone}(S_{n-1}, d_n)$ and let $\pi_n : S_{n+1} \rightarrow S_n$ be the projection map. Define

$$\text{cone}(S, D) = \varprojlim (S_n, \pi_n).$$

Note that if S is a subset of the Hilbert cube, then $\text{cone}(S, D)$ can be naturally embedded into the Hilbert cube as well. Note also that the inverse system in the above definition is increasing. Thus, we treat S_n as a face of $\text{cone}(S, D)$, for each $n \in \mathbb{N}$. Given that, all cone points of the simplices S_n belong to $\text{cone}(S, D)$ and we refer to them as to the *cone points of the iterated cone*. In principle, the iterated cone over D may depend on the ordering of the set D . As we will see, this does not happen.

Lemma 6.5 *Let S be a simplex, $\varepsilon > 0$ and $\varphi : S \rightarrow S$ be an affine homeomorphism. Given two points $s_1, s_2 \in S$ if $d_S(\varphi(s_1), s_2) < \varepsilon$, then there is an affine homeomorphism $\varphi' : \text{cone}(S, s_1) \rightarrow \text{cone}(S, s_2)$ extending φ such that*

$$d_S(\varphi(\pi_1(s)), \pi_2(\varphi'(s))) < \varepsilon$$

for every $s \in \text{cone}(S, s_1)$, where $\pi_1 : \text{cone}(S, s_1) \rightarrow S$ and $\pi_2 : \text{cone}(S, s_2) \rightarrow S$ are the projection maps.

Proof The map φ' is the affine map whose restriction to S is φ and which maps $c(s_1)$ to $c(s_2)$. To see that $d_S(\pi_2(\varphi'(s)), \varphi(\pi_1(s))) < \varepsilon$ for each $s \in \text{cone}(S, s_1)$, note that $\pi_2 \circ \varphi'$ and $\varphi \circ \pi_1$ are two affine continuous maps such that $d_S(\pi_2(\varphi'(e)), \varphi(\pi_1(e))) < \varepsilon$ holds for every extreme point of $\text{cone}(S, s_1)$. Thus, by Proposition 2.1, $d_S(\pi_2(\varphi'(s)), \varphi(\pi_1(s))) < \varepsilon$ is true for every $s \in \text{cone}(S, s_1)$. \square

Proposition 6.6 *Given a simplex S and an affine homeomorphism $\varphi : S \rightarrow S$, if $D = (d_n : n \in \mathbb{N})$ and $E = (e_n : n \in \mathbb{N})$ are two subsets of S*

such that $\text{cl}(E) = \text{cl}(\varphi''D)$, then there is an extension φ' of φ to an affine homeomorphism $\varphi' : \text{cone}(S, D) \rightarrow \text{cone}(S, E)$.

Proof Write $X = \text{cone}(S, D)$ and $Y = \text{cone}(S, E)$ and define X_n and Y_n as $X_0 = Y_0 = S$ and $X_{n+1} = \text{cone}(X_n, d_n)$ and $Y_{n+1} = \text{cone}(Y_n, e_n)$ so that $X = \varprojlim X_n$ and $Y = \varprojlim Y_n$. We define inductively two increasing sequences of natural numbers n_i and m_i and affine continuous maps $\varphi_i : X_{n_i} \rightarrow Y_{m_i}$ and $\psi_i : Y_{m_i} \rightarrow X_{n_{i+1}}$ such that $\varphi_0 = \varphi$ and

- $(\varphi_n, \psi_n : n \in \mathbb{N})$ is an approximate intertwining.
- $\text{rng}(\varphi_n)$ and $\text{rng}(\psi_n)$ is a subcone of X or Y , for each $n \in \mathbb{N}$.

The theorem will then follow by Proposition 4.1.

Let $\varphi_0 = \psi_0^{-1} = \varphi$. Suppose φ_i, ψ_i are defined. We need to define $\varphi_{i+1} : X_{n_{i+1}} \rightarrow Y_{m_{i+1}}$. Let $\varphi'_i = \psi_i^{-1}$ and $X'_{n_i} = \text{rng}(\psi_i) = \text{dom}(\varphi'_i)$. By our assumption, X'_{n_i} is a subcone, so there are $p_1, \dots, p_k \in \mathbb{N}$ such that $X'_{n_i} = \text{cone}(S, d_{p_1}, \dots, d_{p_k})$. Find $n_{i+1} > n_i$ such that $p_j < n_{i+1}$ for each $j \leq k$. Let $\{q_1, \dots, q_l\} = \{1, \dots, n_{i+1}\} \setminus \{p_1, \dots, p_k\}$. By Lemma 6.3, $X_{n_{i+1}}$ is affinely homeomorphic to $\text{cone}(X'_{n_i}, d_{q_1}, \dots, d_{q_l})$. For each $j \leq l$ write $X^j_i = \text{cone}(X'_{n_i}, d_{q_1}, \dots, d_{q_j})$. We inductively find increasing numbers $m^j_i \in \mathbb{N}$ and maps $\varphi^j_i : X^j_i \rightarrow Y_{m^j_i}$ so that $\varphi^0_i = \varphi'_i$, $\varphi^j_i \subseteq \varphi^{j+1}_i$ and for each j we have

$$d_{Y_{m^j_i}}(\varphi^j_i(x \restriction X_{n_i}), \varphi^j_i(x) \restriction Y_{m_i}) < \frac{1}{2^{i+j}} \quad (3)$$

for each $x \in X^j_i$. Given φ^j_i , find m^{j+1}_i such that

$$d_S(\varphi(d_{q_{j+1}}), e_{m^{j+1}_i}) = d_{Y_{m^j_i}}(\varphi^j_i(d_{q_{j+1}}), e_{m^{j+1}_i}) < \frac{1}{2^{i+j}}$$

and use Lemma 6.5 to extend φ^j_i to φ^{j+1}_i so that (3) holds. At the end, put $\varphi_{i+1} = \varphi^l_i$ and $m_{i+1} = m^l_i$. The map ψ_{i+1} is defined analogously.

The condition (3) and an analogous inequality for ψ_i witness that φ_i and ψ_i form an approximate intertwining. \square

Applying Proposition 6.6 to $\varphi = \text{id}$ we get that the definition of $\text{cone}(S, D)$ indeed does not depend on the ordering of D .

Lemma 6.7 *Given a simplex S , with a countable subset D and $d \in S \setminus D$ we have*

$$\text{cone}(S, \{d\} \cup D) \simeq \text{cone}(\text{cone}(S, D), d).$$

Proof Note first that if $(X_n, \pi_n : n \in \mathbb{N})$ is an increasing inverse system of simplices and $x \in X_0$, then $\text{cone}(\varprojlim X_n, x) \simeq \varprojlim \text{cone}(X_n, x)$, where the latter inverse system has the natural projection maps $\pi'_n : \text{cone}(X_n, x) \rightarrow \text{cone}(X_{n-1}, x)$ that extend π_n and map the cone point to the cone point.

Write $D = \{d_1, d_2, \dots\}$ and $X_n = \text{cone}(S, d_1, \dots, d_n)$. By the remark above, $\text{cone}(\text{cone}(S, D), d) \simeq \varprojlim \text{cone}(X_n, d)$ and by Lemma 6.2 we have $\text{cone}(X_n, d) \simeq \text{cone}(S, d, d_1, \dots, d_n)$. Thus, $\text{cone}(\text{cone}(S, D), d)$ is affinely homeomorphic to $\varprojlim (\text{cone}(S, d, d_1, \dots, d_n))$, which is equal to $\text{cone}(S, \{d\} \cup D)$. \square

Now we look at the extreme point of the iterated cones.

Lemma 6.8 *Suppose S is a simplex, $s_1, s_2 \in S$ and let $y \in \text{cone}(S, s_1)$ be such that*

$$y = \alpha x + (1 - \alpha)c(s_1)$$

for some $x \in S$ and $0 < \alpha < 1$. Write $\pi : \text{cone}(\text{cone}(S, s_1), s_2) \rightarrow \text{cone}(S, s_1)$ for the projection map. Then for every $y' \in \text{cone}(\text{cone}(S, s_1), s_2)$ with $\pi(y') = y$ there exists a unique $x' \in \text{cone}(S, s_2)$ such that

$$y' = \alpha x' + (1 - \alpha)c(s_1),$$

and $\pi(x') = x$.

Proof Uniqueness of x' is immediate. Let $c_1 = c(s_1)$ and $c_2 = c(s_2)$. As $y' \in \text{cone}(\text{cone}(S, s_1), s_2)$, there is $z \in \text{cone}(S, s_1)$ and $\delta \in [0, 1]$ such that $y' = \delta z + (1 - \delta)c_2$. As $z \in \text{cone}(S, s_1)$, there is $w \in S$ and $\rho \in [0, 1]$ such that $z = \rho w + (1 - \rho)c_1$.

Note that $0 < \alpha$ implies $y \neq c_1$, and so $\delta\rho + 1 - \delta \neq 0$, for if it were 0, then $\delta = 1, \rho = 0$ and so $y' = z = c_1$, meaning $y = \pi(y') = \pi(c_1) = c_1$. Similarly, $\alpha < 1$ implies $y \notin S$, and so $\delta\rho + 1 - \delta \neq 1$, for if it were 1, then either $\delta = 0$ or $\rho = 1$, and in either case $y' \in \text{cone}(S, s_2)$, meaning $y = \pi(y') \in S$.

Note that

$$\begin{aligned} y' &= \delta z + (1 - \delta)c_2 = \delta\rho w + (\delta - \delta\rho)c_1 + (1 - \delta)c_2 \\ &= (\delta\rho + 1 - \delta) \left[\frac{\delta\rho}{\delta\rho + 1 - \delta} w + \frac{1 - \delta}{\delta\rho + 1 - \delta} c_2 \right] + (\delta - \delta\rho)c_1 \end{aligned}$$

and put $x' = \frac{\delta\rho}{\delta\rho + 1 - \delta} w + \frac{1 - \delta}{\delta\rho + 1 - \delta} c_2$ so that

$$y' = (\delta\rho + 1 - \delta)x' + (\delta - \delta\rho)c_1.$$

Note that $x' \in \text{cone}(S, s_2)$. We claim that $\pi(x') = x$ and $\delta\rho + 1 - \delta = \alpha$. This will imply that $y' = \alpha x' + (1 - \alpha)c_2$, i.e. that x' is as needed.

First, $\pi(x') = \frac{\delta\rho}{\delta\rho+1-\delta}\pi(w) + \frac{1-\delta}{\delta\rho+1-\delta}\pi(c_2)$. But $\pi(w) = w \in S$ and $\pi(c_2) = s_2$, so indeed $\pi(x') \in S$. Now,

$$y = \pi(y') = (\delta\rho + 1 - \delta)\pi(x') + (\delta - \delta\rho)\pi(c_1)$$

and $\pi(c_1) = c_1$. As every element of $\text{cone}(S, s_1) \setminus (S \cup \{c_1\})$ is uniquely determined as a convex combination of c_1 and an element of S , we necessarily have $\pi(x') = x$ and $\delta\rho + 1 - \delta = \alpha$. \square

Lemma 6.9 *Given a simplex S and its countable subset D let $\{v_n : n \in \mathbb{N}\}$ be the set of cone points of $\text{cone}(S, D)$. Then*

$$\text{ext}(\text{cone}(S, D)) = \text{ext}(S) \cup \{v_n : n \in \mathbb{N}\}.$$

Proof By Lemma 6.1, S is a face of $\text{cone}(S, D)$, so $\text{ext}(S) \subseteq \text{ext}(\text{cone}(S, D))$. Also, for each n the simplex $\text{cone}(S, d_0, \dots, d_n)$ is a face of $\text{cone}(S, D)$, which shows that $v_n \in \text{ext}(\text{cone}(S, D))$. We need to show that there are no more extreme points in $\text{cone}(S, D)$. Let $y \in \text{cone}(S, D)$ be such that $y \notin S$ and $y \neq v_n$ for each n . Pick n such that $y \upharpoonright n \notin S$. This means that there exists λ with $0 < \lambda < 1$ and $s \in S$ such that

$$y \upharpoonright n = \lambda s + (1 - \lambda)v_n.$$

By Lemma 6.8, for each $m > n$ there exists $s_m \in \text{cone}(S, d_{n+1}, \dots, d_m)$ such that

$$y \upharpoonright m = \lambda s_m + (1 - \lambda)v_n.$$

Uniqueness of s_m implies that $s_{m_2} \upharpoonright m_1 = s_{m_1}$ for each $m_2 > m_1 > n$. Let $x \in \text{cone}(S, \{d_{n+1}, d_{n+2}, \dots\})$ be such that $x \upharpoonright k = s_k$ for each $k > n$. By Lemma 6.7, we have

$$\text{cone}(S, D) = \text{cone}(\text{cone}(S, \{d_{n+1}, d_{n+2}, \dots\}), d_0, \dots, d_n)$$

and in the latter simplex we have $y = \lambda x + (1 - \lambda)v_n$, which shows that y is not an extreme point of $\text{cone}(S, D)$. \square

Lemma 6.10 *Given a simplex S and its countable subset D , let $\{v_n : n \in \mathbb{N}\}$ be the set of cone points of $\text{cone}(S, D)$. Then each v_n is an isolated point of $\text{ext}(\text{cone}(S, D))$.*

Proof Write $D = \{d_1, d_2, \dots\}$ so that v_n is the cone point over d_n , for each $n \in \mathbb{N}$. We need to show that v_n is isolated from $\text{ext}(S) \cup \{v_i : i \neq n\}$. Write $\pi_n^\infty : \text{cone}(S, D) \rightarrow \text{cone}(S, d_1, \dots, d_n)$ for the projection map. Clearly, $v_n = \pi_n^\infty(v_n)$ has positive distance from $S \cup \{v_i : i < n\}$ in $\text{cone}(S, d_1, \dots, d_n)$, so let U be an open neighborhood of v_n in $\text{cone}(S, d_1, \dots, d_n)$ that is disjoint from S and $\{v_i : i < n\}$. Then $(\pi_n^\infty)^{-1}(U)$ is an open neighborhood of v_n that is disjoint from $\text{ext}(S)$ and does not contain any v_i with $i \neq n$. \square

7 The blow-up construction

Given a countable dense subset D of a metric space (X, δ_X) with $\delta_X < \frac{1}{2}$, enumerate as $(p_n = (x_n, y_n, s_n, t_n) : n \in \mathbb{N})$, with infinite repetitions, all quadruples (x, y, s, t) with $x, y \in D$ distinct and $s < t \in \mathbb{Q} \cap (0, \frac{1}{2})$ such that $s < \delta_X(x, y) < t$. Call the sequence $(p_n : n \in \mathbb{N})$ the *metric scheme* of (X, δ_X, D) and denote it by $\text{Sch}(X, \delta_X, D)$. For every $n \in \mathbb{N}$ and $p_n = (x_n, y_n, s_n, t_n)$ in the metric scheme, write $p_n(1)$ for x_n and $p_n(2)$ for y_n . Write also $p_n(D)$ for the n th element of $\text{Sch}(X, \delta_X, D)$.

Recall that given a simplex S and two points $x_1, x_2 \in S$ the double cone $\text{cone}(S, x_1, x_2)$ is the simplex $\text{cone}(\text{cone}(S, x_1), x_2)$. Now, given a simplex S and its subset $X \subseteq S$ together with a metric d_X on X we will define another, bigger, simplex, which will be used to encode the metric in its affine structure.

Let $(p_n : n \in \mathbb{N})$ enumerate the metric scheme of $(X, \frac{1}{2}d_X)$. Define an increasing inverse system of simplices as follows. Let $B_0 = S$ and $B_n = \text{cone}(B_{n-1}, p_n(1), p_n(2))$. Define the *blow-up* of S with respect to (X, d_X) (slightly abusing the notation), denoted by $B(S, X, d_X)$ as $\varprojlim B_n$ and write $c_1(p_n)$ for the cone point over $p_n(1)$ in $B(S, X, d_X)$ and $c_2(p_n)$ for the cone point over $p_n(2)$ in $B(S, X, d_X)$.

Definition 7.1 Given a closed subspace (X, d_X) of \mathbb{U}_1 with $d_X < 1$, define $B(X, d_X)$ as $B(S_Z(X, d_X), X, d_X)$.

In principle, the blow-up construction depends only on the enumeration (with repetitions) of pairs of points in D . We keep, however, the sequence of p_n 's for reference to the further construction. Note that each pair of distinct points in D appears infinitely often in the sequence. Analogous arguments as in Sect. 6, show that the blow-up depends neither on the ordering p_n 's nor on the choice of the dense set.

Proposition 7.2 Suppose S and T are simplices, $X \subseteq S$ and $Y \subseteq T$ are their subsets, $d_X, d_Y < 1$ are metrics on X and Y , respectively and $D \subseteq X$ and $E \subseteq Y$ are countable dense sets. Given an affine homeomorphism $\varphi : S \rightarrow T$ such that $\varphi''X = Y$ and $\varphi|_X$ is an isometry of (X, d_X) and (Y, d_Y) , there is an affine homeomorphism $\bar{\varphi} : B(S, X, d_X) \rightarrow B(T, Y, d_Y)$ such that $\bar{\varphi}$ extends φ and

- (a) for every $p \in \text{Sch}(X, \frac{1}{2}d_X, D)$ there exists $q \in \text{Sch}(Y, \frac{1}{2}d_Y, E)$ such that if $p = (x_1, x_2, s, t)$, then $q = (y_1, y_2, s, t)$ and $\bar{\varphi}$ maps $c_1(p)$ to $c_1(q)$ and $c_2(p)$ to $c_2(q)$,
- (b) for every $q \in \text{Sch}(Y, \frac{1}{2}d_Y, E)$ there exists $p \in \text{Sch}(X, \frac{1}{2}d_X, D)$ such that if $q = (y_1, y_2, s, t)$, then $p = (x_1, x_2, s, t)$ and $\bar{\varphi}^{-1}$ maps $c_1(q)$ to $c_1(p)$ and $c_2(q)$ to $c_2(p)$.

Proof The proof is essentially the same as that of Proposition 6.6 and we only sketch it. Write B_n for the simplices in the inverse system of $B(S, X, d_X)$ and C_n for the simplices in the inverse system of $B(T, Y, d_Y)$. By induction on $i \in \mathbb{N}$, construct an approximate intertwining $\varphi_i : B_{n_i} \rightarrow C_{m_i}$ and $\psi_i : C_{m_i} \rightarrow B_{n_{i+1}}$ so that $\varphi_0 = \varphi$. At the inductive construction, when extending a map from B_{n_i} to $B_{n_{i+1}}$, make sure that if $p_n(D) = (x_1, x_2, s, t)$, then for some $m \in \mathbb{N}$ with $p_m(E) = (y_1, y_2, s, t)$, the point $c_1(p_n(D))$ is mapped to $c_1(p_m(E))$, $c_2(p_n(D))$ is mapped to $c_2(p_m(E))$ so that

$$d_T(\varphi(x_1), y_1), d_T(\varphi(x_2), y_2) < 2^{-m} \quad (4)$$

and $d_Y(y_1, y_2) \in (s, t)$. Analogous conditions apply when we extend a map from C_{m_i} to $C_{m_{i+1}}$. The fact that the above is possible follows at once from the fact that φ was an isometry and the elements in the schemes are enumerated with infinite repetitions.

Once the construction is finished, (4) and the symmetric condition involving d_S imply that the sequence forms an approximate intertwining. Then, an application of Proposition 4.1 gives an affine homeomorphism $\bar{\varphi} : \varprojlim B_i \rightarrow \varprojlim C_i$. The conditions (a) and (b) follow directly from the construction. \square

8 Coding a metric into the affine structure of a simplex

Given a metric space (X, d_X) with $d_X < 1$, which we assume is a subspace of the Urysohn sphere \mathbb{U}_1 , and a dense countable subset D of X , let $(p_n = ((x_n, y_n, s_n, t_n) : n \in \mathbb{N}))$ be the associated metric scheme of $(X, \frac{1}{2}d_X)$. Consider the blow-up $B(X, d_X)$ and let $M(X, d_X, D)$ be the subset of $B(X, d_X)$ consisting of the points

$$qc_1(p_n) + (1 - q)c_2(p_n) \quad \text{and} \quad qc_2(p_n) + (1 - q)c_1(p_n)$$

for $q \in (s_n, t_n) \cap \mathbb{Q}$ (note that the points $c_1(p_n)$ and $c_2(p_n)$ do not belong to the closure of this set since $(s_n, t_n) \cap \mathbb{Q}$ is isolated away from 0).

Use the Kuratowski–Ryll–Nardzewski theorem [46, Theorem 12.13], to find a countable dense subset $D(X)$ for every (nonempty) closed subset X of \mathbb{U}_1 so that the map $D : X \mapsto D(X)$ is Borel. Define now $\Phi(X, d_X)$ as $\text{cone}(B(X, d_X), M(X, d_X, D(X)))$. It is easy to check that Φ is a Borel map

from the space of closed (nonempty) subsets of \mathbb{U}_1 to the space of separable Choquet simplices (cf. Sect. 2 for the Borel structure on the space of simplices). In fact, the map $(X, D) \mapsto \text{cone}(B(X, d_X), M(X, d_X, D))$, which maps a pair of a Polish space and its dense countable subset to a simplex, is continuous in the following natural topologies. First, any such pair (X, D) with $X \subseteq \mathbb{U}_1$ is uniquely determined by D (as $X = \overline{D}$) and hence the topology on such pairs can be taken as the topology on the space on sequences of elements of \mathbb{U}_1 , which is the product topology on $(\mathbb{U}_1)^{\mathbb{N}}$. Second, the simplices $\text{cone}(B(X, d_X), M(X, d_X, D))$ obtained from the pairs (X, D) are naturally embedded into the Hilbert cube $[0, 1]^{\mathbb{N}}$, and the natural topology there is the Vietoris topology on compact subsets of the Hilbert cube.

Proposition 8.1 *Suppose (X, d_X) and (Y, d_Y) are subspaces of \mathbb{U}_1 with $d_X, d_Y < 1$. If (X, d_X) and (Y, d_Y) are isometric, then the simplices $\Phi(X, d_X)$ and $\Phi(Y, d_Y)$ are affinely homeomorphic.*

Proof For simplicity, identify X with $Z(X)$ and Y with $Z(Y)$ (see Proposition 3.2) and let $\varphi : \mathbb{U}_1 \rightarrow \mathbb{U}_1$ be an isometry such that $\varphi''X = Y$. By Propositions 5.21 and 7.2, there is an affine homeomorphism $\bar{\varphi} : B(X, d_X) \rightarrow B(Y, d_Y)$ which extends φ and such that

- for each $p = (x_1, x_2, s, t)$ in the scheme of $(X, \frac{1}{2}d_X)$ there is $q = (y_1, y_2, s, t)$ in the scheme of $(Y, \frac{1}{2}d_Y)$ such that $\bar{\varphi}$ maps the cone points $c_1(p)$ to $c_1(q)$ and $c_2(p)$ to $c_2(q)$,
- for each $q = (y_1, y_2, s, t)$ in the scheme of $(Y, \frac{1}{2}d_Y)$ there is $p = (x_1, x_2, s, t)$ in the scheme of $(X, \frac{1}{2}d_X)$ such that $\bar{\varphi}^{-1}$ maps $c_1(q)$ to $c_1(p)$ and $c_2(q)$ to $c_2(p)$.

The above imply that $\bar{\varphi}''\text{cl}(M(X, d_X, D(X))) = \text{cl}(M(Y, d_Y, D(Y)))$ and hence, by Proposition 6.6, there is an affine homeomorphism $\bar{\varphi} : \Phi(X, d_X) \rightarrow \Phi(Y, d_Y)$ that extends $\bar{\varphi}$. \square

Proposition 8.1 shows that Φ maps isometric metric spaces to affinely homeomorphic simplices. To show that Φ is a reduction, we will restrict attention to perfect separable metric spaces.

Claim 8.2 *If (X, d_X) is perfect with $d_X < 1$, then the set of nonisolated extreme points of $\Phi(X, d_X)$ is equal to $\text{ext}(S_Z(X, d_X))$.*

Proof X is perfect and dense in $\text{ext}(S_Z(X, d_X))$ by Corollary 5.20, so all extreme points of $S_Z(X, d_X)$ are nonisolated in $\text{ext}(S_Z(X, d_X))$. Since the simplex $S_Z(X, d_X)$ is a face of $\Phi(X, d_X)$, the extreme points of $S_Z(X, d_X)$ are still nonisolated extreme points of $\Phi(X, d_X)$. Now, Lemmas 6.9 and 6.10 and analogous statements for the blow-up (here we use the fact that $[s, t] \subseteq (0, \frac{1}{2})$ for $s < t$ appearing in the metric scheme of $(X, \frac{1}{2}d_X)$ in order to ensure that

$c_1(p_n)$ and $c_2(p_n)$ remain isolated after passing from $B(X, d_X)$ to $\Phi(X, d_X)$) imply that all the other extreme points of $\Phi(X, d_X)$ are isolated. \square

Lemma 8.3 *Suppose (X, d_X) is a perfect metric subspace of \mathbb{U}_1 . If $x, y \in X$ and $x \neq y$, then $\frac{1}{2}d_X(x, y)$ is the only $\alpha \in (0, \frac{1}{2})$ such that*

$$\begin{aligned} & \forall s < t \in \mathbb{Q} \cap \left(0, \frac{1}{2}\right) \text{ with } \alpha \in (s, t) \\ & \exists V_x, V_y \subseteq \Phi(X, d_X) \text{ open neighborhoods of } x \text{ and } y \text{ in } \Phi(X, d_X) \\ & \forall c_1 \in V_x \forall c_2 \in V_y \text{ isolated extreme points} \\ & \left[(\exists \lambda \in (0, \frac{1}{2}) \quad \lambda c_1 + (1 - \lambda)c_2 \right. \\ & \quad \text{is a limit of isolated extreme points of } \Phi(X, d_X)) \\ & \Rightarrow (\exists \lambda \in (s, t) \quad \lambda c_1 + (1 - \lambda)c_2 \\ & \quad \text{is a limit of isolated extreme points of } \Phi(X, d_X)) \Big]. \end{aligned} \quad (5)$$

Proof Write D for the countable dense subset $D(X)$ of X . By Claim 8.2 and Lemma 6.9, the only isolated extreme points of $\Phi(X, d_X)$ are in the sets $E_1 = \{c_1(p), c_2(p) : p \in \text{Sch}(X, \frac{1}{2}d_X, D)\}$ and $E_2 = \{c(z) : z \in M(X, d_X, D)\}$. It follows from the iterated cone construction that if $z \in M(X, d_X, D)$, then for any isolated extreme point e of $\Phi(X, d_X)$, no point in the set $\{\lambda c(z) + (1 - \lambda)e : \lambda \in [0, 1]\}$ is a limit of isolated extreme points of $\Phi(X, d_X)$. On the other hand, if $e_1, e_2 \in E_1$ are such that for some $\lambda \in [0, 1]$ the point $\lambda e_1 + (1 - \lambda)e_2$ is a limit of isolated extreme points of $\Phi(X, d_X)$, then there is $p \in \text{Sch}(X, \frac{1}{2}d_X, D)$ such that $\{e_1, e_2\} = \{c_1(p), c_2(p)\}$. The latter follows from the fact that the intersection of $B(X, d_X)$ with the closure of the set $\{c(z) : z \in M(X, d_X, D)\}$ is exactly the closure of $M(X, d_X, D)$.

Pick $x \neq y \in X$. We claim that $\alpha = \frac{1}{2}d_X(x, y)$ satisfies the condition (5). Pick any $(s, t) \subseteq (0, \frac{1}{2})$ basic open neighborhood of α and let $\varepsilon > 0$ be such that $(\alpha - \varepsilon, \alpha + \varepsilon) \subseteq (s, t)$. Since X is a topological subspace of $S(X, d_X)$, there are V_x^1 and V_y^1 open neighborhoods of x and y in $S(X, d_X)$, respectively, such that $V_x^1 \cap X \subseteq \text{ball}_{(X, d_X)}(x, \varepsilon)$ and $V_y^1 \cap X \subseteq \text{ball}_{(X, d_X)}(y, \varepsilon)$. Let V_x and V_y be open neighborhoods of x and y in $\Phi(X, d_X)$ such that if $c_1 \in V_x$ and $c_2 \in V_y$, then $\pi(c_1) \in V_x^1$ and $\pi(c_2) \in V_y^1$, where $\pi : \Phi(X, d_X) \rightarrow S(X, d_X)$ denotes the projection map. We claim that V_x and V_y are as needed. Let $c_1 \in V_x$ and $c_2 \in V_y$ be arbitrary extreme isolated points such that for some $\lambda \in (0, \frac{1}{2})$ the point $\lambda c_1 + (1 - \lambda)c_2$ is a limit of isolated extreme points of $\Phi(X, d_X)$. By the remarks in the previous paragraph, there is $p \in \text{Sch}(X, \frac{1}{2}d_X, D)$ such that $p = (d_1, d_2, q, r)$ for some $d_1, d_2 \in D$ with $\{\pi(c_1), \pi(c_2)\} = \{d_1, d_2\}$ and $(q, r) \subseteq (0, \frac{1}{2})$ is a basic open neighborhood of $\frac{1}{2}d_X(d_1, d_2)$. Now, since $c_1 \in V_x$ and $c_2 \in V_y$ we have that $\frac{1}{2}d_X(d_1, d_2) \in (\alpha - \varepsilon, \alpha + \varepsilon) \subseteq (s, t)$,

so $(q, r) \cap (s, t)$ is nonempty. Pick any $\lambda \in (q, r) \cap (s, t)$ and note that in either case: $(d_1, d_2) = (\pi(c_1), \pi(c_2))$ or $(d_1, d_2) = (\pi(c_2), \pi(c_1))$, the point $\lambda c_1 + (1 - \lambda)c_2$ is also a limit of isolated extreme points of $\Phi(X, d_X)$ by the definition of $M(X, d_X, D)$ and the fact that $\lambda \in (q, r)$.

We also need to show that α is the only number in $(0, \frac{1}{2})$ satisfying (5) for x and y . Pick any $\beta \in (0, \frac{1}{2})$ distinct from α . Let $\varepsilon > 0$ be smaller than $|\alpha - \beta|$. Pick a basic open neighborhood $(s, t) \subseteq (\beta - \varepsilon/2, \beta + \varepsilon/2)$ of β . We claim that (s, t) witnesses that (5) is not satisfied. Let V_x and V_y be arbitrary open neighborhoods of x and y , respectively, in $\Phi(X, d_X)$. Pick $d_1, d_2 \in D$ such that $d_X(d_1, x) < \varepsilon/2$, $d_X(d_2, y) < \varepsilon/2$ and $d_1 \in V_x$, $d_2 \in V_y$. Note that $\frac{1}{2}d_X(d_1, d_2) \in (\alpha - \varepsilon/2, \alpha + \varepsilon/2)$ and that (s, t) is disjoint from $(\alpha - \varepsilon/2, \alpha + \varepsilon/2)$. Find a basic open neighborhood (q, r) of $\frac{1}{2}d_X(d_1, d_2)$ in $(0, \frac{1}{2})$ such that $[q, r] \cap (s, t) = \emptyset$. Find $n \in \mathbb{N}$ big enough so that $p_n \in \text{Sch}(X, \frac{1}{2}d_X, D)$ is equal to (d_1, d_2, q, r) and such that letting $c_1 = c_1(p_n)$, $c_2 = c_2(p_n)$ we have $c_1 \in V_x$ and $c_2 \in V_y$. Now, there is $\lambda \in (0, \frac{1}{2})$ such that $\lambda c_1 + (1 - \lambda)c_2$ is a limit of isolated extreme points of $\Phi(X, d_X)$ but the set of such λ is equal to $[q, r]$, which is disjoint from (s, t) . This shows that β does not satisfy (5) and ends the proof. \square

Now we are ready to finish the proof that Φ is a reduction.

Proposition 8.4 *If (X, d_X) and (Y, d_Y) are perfect closed subspaces of \mathbb{U}_1 with $d_X, d_Y < 1$ and $\Phi(X, d_X)$ is affinely homeomorphic to $\Phi(Y, d_Y)$, then (X, d_X) is isometric to (Y, d_Y) .*

Proof Let $\varphi : \Phi(X, d_X) \rightarrow \Phi(Y, d_Y)$ be an affine homeomorphism. Note that φ maps nonisolated extreme points of $\Phi(X, d_X)$ to nonisolated extreme points of $\Phi(Y, d_Y)$, so, by Claim 8.2, $\varphi''\text{ext}(S(X, d_X)) = \text{ext}(S(Y, d_Y))$. Now, $X \subseteq S(X, d_X)$ and $Y \subseteq S(Y, d_Y)$ are dense G_δ sets, so there is a comeager set $X' \subseteq X$ such that $Y' = \varphi''X'$ is comeager in Y . Note that since φ preserves the topological and affine structure, (5) is preserved by φ and so Lemma 8.3 implies that for $x_1, x_2 \in X'$ we have $d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2)$. This means that (X', d_X) and (Y', d_Y) are isometric. Since X' is comeager in X and Y' is comeager in Y , the spaces X and Y are isometric as well. \square

Theorem 1.4 now follows from Propositions 3.1, 8.1 and 8.4.

9 Concluding remarks and open questions

As we have mentioned in the introduction, the theory of functorial reductions is still to be developed in the context of descriptive set-theoretic complexity theory. One way to make it precise can be made in terms of Polish groupoids. In this setting, a *functorial reduction* would be a full (Borel) functor between

Polish groupoids. It is not difficult to check that our reduction of the isometry of complete separable metric spaces to the affine homeomorphism of Choquet simplices is in fact functorial in the above sense. Another notion of functoriality can be also defined in terms of functors on small categories in order to capture all morphisms between objects. This setting is more subtle.

Another major problem that is still left open is that of the complexity of the homeomorphism relation for compact subspaces of $[0, 1]^2$. It is known that the homeomorphism relation for compact subspaces of the interval is classifiable by countable structures but the homeomorphism relation in two dimensions is turbulent as shown by Hjorth [40].

Question 9.1 Is the relation of homeomorphism of compact subsets of $[0, 1]^2$ (or two-dimensional compact spaces) a complete orbit equivalence relation?

In [28, Theorem 7.3] Farah, Toms and Törnquist showed that the isomorphism of separable simple nuclear C^* -algebras is below an action of the automorphism group of the Cuntz algebra $\text{Aut}(\mathcal{O}_2)$. Since complete orbit equivalence relations are typically induced by actions of universal Polish groups, the following question is natural.

Question 9.2 Is the group $\text{Aut}(\mathcal{O}_2)$ a universal Polish group?

On the other hand, the following seems to be unknown:

Question 9.3 Can the group $U(H)$ (the unitary group of the infinite-dimensional separable Hilbert space) induce a complete orbit equivalence relation?

Finally, this paper reveals an interesting connection between three categories of objects: separable metric spaces, metrizable Choquet simplices and separable Banach spaces. It is somewhat parallel to the connection between metric spaces and Banach spaces given by the Arens–Eells extensions. Recall that given a separable metric space, its Arens–Eells extension is a separable Banach space and the assignment of the Arens–Eells extensions is isometry invariant. On the other hand, given a simplex, one can look at the space of affine continuous functions on it. Now, the S -extensions give an invariant assignment of separable Banach spaces to separable metric spaces that factors through simplices as follows:

$$\text{metric space} \xrightarrow{S\text{-extension}} \text{Choquet simplex} \xrightarrow{\text{affine space}} \text{Banach space}$$

The three categories of separable metric spaces, metrizable Choquet simplices and separable Banach spaces contain universal objects that have been studied independently. While the universal objects in the first two classes are the

Urysohn space and the Poulsen simplex, the category of separable Banach spaces contains the Gurarii space [37], which is a separable Banach space with an almost isometric extension property. This space turns out to be unique [57] (see also [51] for a recent proof of this result). It is known that the Gurarii space is isomorphic to the space of affine functions on the Poulsen simplex [57] but the relations between the Urysohn space and the Gurarii space or the Poulsen simplex remain unclear. Fonf and Wojtaszczyk [30] showed that the Gurarii space is not isomorphic to the Arens–Eells extension of the Urysohn space. While we think that the S -extension of the Urysohn space is not affinely homeomorphic to the Poulsen simplex, we suspect that the compact convex set $S'(\mathbb{U}_1)$ is the Poulsen simplex (see page 21 for the definition).

Question 9.4 Is $S'(X)$ always a simplex for a separable metric space X ? Is $S'(\mathbb{U}_1)$ affinely homeomorphic to the Poulsen simplex?

Acknowledgments Part of this work was done during the author's stay at the Fields Institute during the thematic program in the fall 2012. The author is grateful for the hospitality of the Fields Institute and would like to thank Antonio Avilés, George Elliott, Ilijas Farah and Stevo Todorčević for many inspiring discussions. The author would also like to thank the anonymous Referee for a number of useful suggestions that improved the quality of this paper.

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